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THE SPACE-TIME MANIFOLD OF RELATIVITY. THE NON-EUCLIDEAN GEOMETRY OF MECHANICS AND ELECTROMAGNETICS.

BY EDWIN B. WILSON AND GILBERT N. LEWIS.

Introduction.

1. The concept of space has different meanings to different persons according to their experience in abstract reasoning. On the one hand is the common space, which for the educated person has been formulated in the three dimensional geometry of Euclid. On the other hand the mathematician has become accustomed to extend the concept of space to any manifold of which the properties are completely determined, as in Euclidean geometry, by a system of self-consistent postulates. Most of these highly ingenious geometries cannot be expected to be of service in the discussion of physical phenomena.

Until recently the physicist has found the three dimensional space of Euclid entirely adequate to his needs, and has therefore been inclined to attribute to it a certain reality. It is, however, inconsistent with the philosophic spirit of our time to draw a sharp distinction between that which is real and that which is convenient,[1] and it would be dogmatic to assert that no discoveries of physics might render so convenient as to be almost imperative the modification or extension of our present system of geometry. Indeed it seemed to Minkowski that such a change was already necessitated by the facts which led to the formulation of the Principle of Relativity.

2. The possibility of associating three dimensional space and one dimensional time to form a four dimensional manifold has doubtless occurred to many; but as long as space and time were assumed to be wholly independent, such a union seemed purely artificial. The idea of abandoning once for all this assumption of independence, although fore-shadowed in Lorentz's use of local time, was first clearly stated by

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[1] See, for example, H. Poincaré, La Science et l'Hypothèse.

Einstein. The theorems of the principle of relativity which correlate space and time appeared, however, far less bizarre and unnatural when Minkowski showed that they were merely theorems in a four dimensional geometry.

Suppose that a student of ordinary space, habituated to the interpretation of geometry with the aid of a definite horizontal plane and vertical axis, should suddenly discover that all the essential geometrical properties of interest to him could be expressed by reference to a new plane, inclined to the horizontal, and a new axis inclined to the vertical. Whereas formerly he had attributed special significance to heights on the one hand and to horizontal extension on the other, he would now recognize that these were purely conventional and that the fundamental properties were those such as distance and angle, which remain invariant in the change to a new system of reference.

Let us now consider a four dimensional manifold formed by adjoining to the familiar x, y, z axes of space a t axis of time. Any point in this manifold will represent a definite place at a definite time. Space then appears as a sort of cross section through this manifold, comprising all points of a given time. For convenience we may temporarily ignore one of the dimensions of space, say z, and discuss the three dimensional manifold of x, y, t. This means that we will consider only positions and motions in a plane. The locus in time of a particle which does not change its position in space, that is, of a particle at rest, will be a straight line parallel to the t axis. Uniform rectilinear motion of a particle will then be represented by a straight line inclined to the t axis.

3. If we adopt the view that uniform motion is only relative, we may with equal right consider the second particle at rest and the first particle in motion. In this case the locus of the second particle must be taken as a new time axis. What corresponding change this will necessitate in our spacial system of reference will depend entirely upon the kind of geometry that we are led to adopt in order to make the geometrical invariants of the transformation correspond to the fundamental physical invariants whose occurrence in mechanics and electromagnetics has led to the principle of relativity.

It is immediately evident that if uniform motion is to be represented by straight lines, the statement that all motion is relative shows that the transformation must be of such a character as to carry straight lines into straight lines. In other words, the transformation must be linear. Further we must assume that the origin of our space and time axes is entirely arbitrary.

The further characteristics of this transformation must be determined by a study of the important physical invariants. Fundamental among these invariants is the velocity of light, which by the second postulate of the principle of relativity must be the same to all observers. Hence any line in our four dimensional manifold which represents motion with the velocity of light must bear the same relation to every set of reference axes. This is a condition which certainly cannot be fulfilled by any transformation of axes to which we are accustomed in real Euclidean space. It is indeed a condition sufficient to determine the properties of that non-Euclidean geometry which we are to investigate.

Minkowski, in his two papers on relativity,[2] used two different methods. In his first and elaborate treatment of the subject he introduced the imaginary unit $\sqrt{-1}$ in such a way that the lines which represent motion with the velocity of light become the imaginary invariant lines familiar to mathematicians who discuss the real and imaginary geometry of Euclidean space. In this way, however, the points of the manifold which represent a particle in position and time become imaginary; the transformations are imaginary; the whole method becomes chiefly analytical. In his second, a brief paper, Minkowski makes use of certain geometrical constructions which have their simplest interpretation only in a non-Euclidean geometry.

4. It is the purpose of the present work to develop the four dimensional non-Euclidean geometry which is demanded by the principle of relativity, and to show that the laws of electromagnetics and mechanics not only can be simply interpreted in this way but also are for the most part mere theorems in this geometry.

In the first sections we shall develop in some detail the non-Euclidean geometry in two dimensions. For it is only by a thorough comprehension of this simpler case that it is possible to proceed into the more difficult domains involving three and four dimensions. This part of the paper will be continued by a discussion of vectors and the vector notation that will be employed. At this point it is possible in a few simple cases to show the applications of the non-Euclidean geometry to problems in kinematics and mechanics.

The sections devoted to three dimensions will be occupied largely with numerous analytical developments of the vector algebra, many of which are directly applicable not only in space of higher dimensions but also in Euclidean space. We are led further to a consideration of certain vectors of singular character. The study of the singular plane leads to the brief consideration of another interesting and important non-Euclidean plane geometry.

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[2] Gesammelte Abhandlungen von Hermann Minkowski, Vol. 2, pp. 352–404 and pp. 431–444.

Passing to the general case of four dimensions we shall meet further new types of vectors, and shall attempt even here to facilitate as far as is possible the visualization of the geometrical results. We shall continue further the analytical development, and in particular consider the properties of the differential operator *quad.* In this connection a very general and important equation for the transformation of integrals is obtained. The idea of the geometric vector field will then be introduced, and the properties of these fields will be taken up in detail.

The subject of electromagnetics and mechanics is prefaced with a short discussion of the possibility of replacing conceptually continuous and discontinuous distributions by one another, and we shall point out that in one important case such a transformation is impossible. The science of electromagnetics is treated both from the point of view of the point charge and from that of the continuous distribution. In both cases it is shown that the field of potential and the field of force are merely the geometrical fields previously mentioned, except for a constant multiplier. Particular attention is given to the field of an accelerated electron,[3] and in this field we find that the vectors of singular properties play an important rôle. With the aid of these vectors the problem of electromagnetic energy is discussed. The science of mechanics, which is treated in a fragmentary way in some preceding sections, is now given a more general treatment, and the conservation laws of momentum, mass and energy are shown to be special deductions from a single general law stating the constancy of a certain four dimensional vector, which we have called the vector of extended momentum. Finally it is pointed out that this last vector gives rise to geometric vector fields which can be identified with the fields of gravitational potential and gravitational force. Moreover, it is shown that these fields are identical in mathematical form with the electromagnetic fields, and that all the equations of the electromagnetic field must be directly applicable to the gravitational.

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[3] There seems to be a widespread impression that the principle of relativity is inadequate to deal with problems involving acceleration. But the essential idea of relativity can be expressed by the statement that there are certain vectors in the geometry of four dimensions which are independent of any arbitrary choice of the axes of space and time. Those problems which involve acceleration will be shown to possess no greater inherent difficulties than those that involve only uniform motion. It is, moreover, especially to be emphasized that the methods which are to be employed in this paper necessitate none of the approximations that are commonly employed in electromagnetic theory. Such terms as "quasi-stationary," for example, will not be used.

In an appendix a few rules for the use of Gibbs's dyadics, which have occasionally been employed in the text, are stated. And a brief discussion of some of the mathematical aspects of our plane non-Euclidean geometry is given.

The Non-Euclidean Geometry in Two Dimensions.

Translation or the Parallel Transformation.

5. In discussing a non-Euclidean geometry various methods of procedure are available; a set of postulates may be laid down, or the differential method of Riemann may be followed, or the theory of groups may be used as by Lie, or (if the geometry falls under the general projective type, as is here the case) the projective measure of length and angle may be made the basis. For our present purpose we need not restrict ourselves to any one of these; but since the first is familiar to all, we shall employ it as far as convenience permits. Some of the other methods will, however, be briefly discussed in the appendix, §§ 64, 65.

With a view to simplicity we shall at first limit the discussion to the case of a plane. Points and lines will be taken as undefined, and most of the relations connecting them will be the same as in Euclidean plane geometry. Thus:[4]

1°. Through two points one and only one line can be drawn.

2°. Two lines intersect in one and only one point, except that

3°. Through any point not on a given line one and only one parallel (non-intersecting) line can be drawn.

4°. The line shall be regarded as a continuous array of points in open order.

6. In regard to congruence or "free mobility" it is important to proceed more circumspectly than did Euclid. The transformations of Euclidean geometry may be divided into translations and rotations, of which the former alone are the same for our geometry. It seems desirable, therefore, to discuss first and in some detail the postulates and propositions relating to this type of transformation, and common to the two geometries. We therefore postulate for translation:

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[4] We make no claim of completeness or independence for these postulates, which are designed primarily to show the points of similarity or dissimilarity between our geometry and the Euclidean. A like remark may be made with respect to proofs of theorems.

5°. Any point P can be carried into any point P', and any two translations which carry P into P' are identical.

6°. Any line is carried into a parallel line.

7°. Any line parallel to PP' remains unchanged.

8°. The succession of two translations is a translation.

These postulates determine the characteristics of a group of geometries of which the two most important are Euclidean geometry and that non-Euclidean geometry with which we are here concerned. Another non-Euclidean geometry belonging to this same group will be discussed briefly in § 31. This group excludes such geometries as the Lobachewskian and the Riemannian in which a parallel to a given line at a given point is not uniquely defined. We shall first proceed to develop some of those general theorems which are true in this whole group of geometries.

I. If two intersecting lines are parallel respectively to two other intersecting lines, the corresponding angles [5] are congruent.

For by translation the points of intersection may be made to coincide, and the lines of the first pair, remaining parallel with the lines of the other pair (6°), must come into coincidence with them, by postulate 3°.

II. The opposite sides of a parallelogram are congruent.

For if $ABCD$ is a parallelogram and if A be translated to B, the line of DC remains unchanged, by 7°, and the line of AD falls along the line of BC by I. Hence D falls on C by 2°.

Cor. If two points P, P' are carried by a translation into Q, Q', the figure $PP'\,Q'\,Q$ is a parallelogram.

7. We may now set up a system of measurement along any line and hence along the whole set of parallel lines. Consider the segment PP'. By the translation which carries P into P', the point P' is carried into a point P'' of the same line. The measure of the separation of P and P' we will call the *interval* [6] PP'. And since the segment PP' is congruent to the segment $P'\,P''$, the intervals PP' and $P'\,P''$ are said to be equal. We may thus mark off any number of equal intervals along the line. We shall assume further the Archimedean postulate.

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[5] The word angle here refers to a geometrical figure only, and does not as yet imply any measure of angle.

[6] We use the word interval to avoid all ambiguity. The notion of distance will be separately considered in Appendix, § 65.

9°. If a sufficient number of equal intervals be laid off on a line, any point of the line may be surpassed.

Now the whole theory of commensurability or incommensurability of two intervals along the same line or parallel lines may be treated by the usual methods. Thus the intervals along a line, starting from any origin upon the line, may be brought into one-to-one correspondence with the series of real numbers. It is, however, to be especially emphasized that we have not established, and cannot establish by the translation alone, any comparison between intervals on non-parallel lines.

III. The diagonals of a parallelogram bisect each other.[7]

For let (Figure 1) the parallelogram *ABCD*, of which the diagonals intersect at *E*, be translated into the position *BB′ C′ C* (by translating *A* to *B*), in which the diagonals intersect at *E′*. Now *BE′* is parallel to *EC*, and *EB* to *CE′*. Hence *BE′* which is congruent to *AE*, is congruent to *EC* by II. Consequently *AE* is congruent to *EC* by 8°.

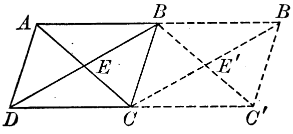

FIGURE 1.

IV. If two triangles have the sides of one respectively parallel to the sides of the other, and if one side of one is congruent to one side of the other, then the remaining sides of the one are respectively congruent to the remaining sides of the other.

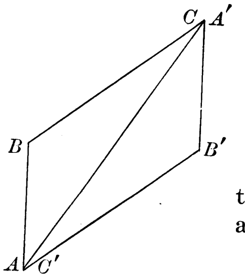

FIGURE 2.

For if the two congruent sides are brought into coincidence by translation, the two triangles will either coincide throughout, or will together (Figure 2) form a parallelogram (II).

Two triangles with the sides of one respectively parallel to the sides of the other will be called similar.

V. In two similar triangles the sides of the one are respectively proportional to the sides of the other.

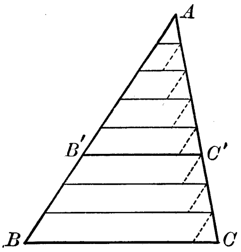

FIGURE 3.

For if *ABC* and *A′B′C′* are the triangles, the vertex *A′* may be made to coincide with *A* by a translation (Figure 3). Suppose, now, that AB' falls along AB, and AC' along AC. Assume that AC and AC' are commensurable. Apply the common measure to the side AC, and through the points of division draw lines parallel to BC and to AB. In the small triangles thus formed the parallel sides will be equal by IV, and therefore the intervals cut off on AB must be equal by II. In case of incommensurability the method of limits may be applied.[8] The case in which the two triangles fall on opposite sides of the common vertex may be treated in a similar manner by the aid of IV.

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[7] Theorems like this and the preceding and some which are to follow are proved in elementary geometries by the aid of propositions (on congruence of triangles) not deducible from translations alone.

8. For our future needs, the conception and the measure of area are fundamental, and it is important to show that this subject may be satisfactorily treated with the aid of the parallel-transformation (that is, the translation) alone. Indeed, any arbitrarily chosen unit intervals along any selected pair of intersecting lines determine a parallelogram which may be taken as having a unit area. By ruling the parallelogram into equal parallelograms by lines parallel to its sides, an arbitrarily small element of area may be obtained. The area enclosed by any curve may be divided into like elements by similar rulings, and thus by the method of limits the enclosed area may be compared with the assumed unit area.[9] In particular some simple propositions on areas will now be deduced.

VI. Any parallelogram with sides parallel to those of the unit parallelogram has an area equal to the product of the intervals along two intersecting sides.

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[8] It may be observed at this point that if two intersecting lines be taken as axes of reference, if systems of measurement (as yet necessarily independent) be set up along the two lines with the point of intersection as common origin, and if to each point P of the plane are assigned coordinates (x, y) equal to the intercepts cut off from the axes by lines through P parallel to the axes, then straight lines are represented by linear equations, and conversely. For the deduction of the equation of a line depends merely upon the properties of triangles similar in our sense. The transformation from any such set of axis to any other such set will clearly be linear.

[9] If axes be introduced as above, the area of a triangle and the area of any closed curve are expressed analytically by the usual formulas.

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \text{and} \qquad \iint dxdy = \int_{\circ} xdy = -\int_{\circ} ydx,$$

in terms of our assumed unit parallelogram. The theorems on areas could then be proved analytically, but the elementary geometric demonstrations seem preferable. It is important to observe further that in a transformation to new axes, such that

$$x = ax' + by' + c, \qquad y = a'x' + b'y' + c',$$

VII. The diagonal of a parallelogram divides it into two equal areas.

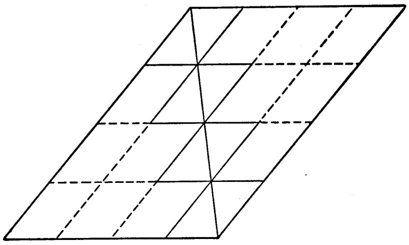

FIGURE 4.

For if the sides of the parallelogram be divided by repeated bisection into 2^n parts, there will be an equal number of equal parallelograms on each side of the diagonal (Figure 4), and in the limit the total area of these parallelograms approaches the area of the triangles.

VIII. If from any point in the diagonal of a parallelogram lines be drawn parallel to the sides, the two parallelograms formed on either side of the diagonal are equal in area (Figure 5).

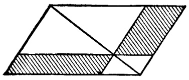

FIGURE 5.

IX. Two parallelograms between the same parallel lines and with congruent bases are equal in area.

Cor. Two triangles having congruent bases on one line and vertices on a parallel line have equal areas.

Cor. The diagonals divide a parallelogram into four equal triangular areas.

Proofs may be given by obvious and familiar methods.

X. Of all parallelograms having two sides common to two sides of a given triangle and a vertex on the third side of the triangle, that one has the greatest area whose vertex bisects that third side.

For in the figure (Figure 6), where ABC is the triangle and E is the middle point of the third side, the difference of the two parallelograms is

$$HBFE - IBGD = MGFE - IHMD = KMEL - IHMD$$
$$= KMEL - KDNL = DMEN.$$

Propositions IV and VIII are used in the proof.

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the value of the area, in terms of the area measured with reference to the new axes, is

$$dxdy = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} dx'dy.'$$

Hence if the measure of area is to be the same, that is, if the unit parallelogram on the new axes is to have a unit area referred to the old axes, the determinant of the transformation must be unity. This implies a relation between the choice of unit intervals on the new axes. Indeed when the unit interval on one of the new axes has been arbitrarily chosen, the unit interval on the other is determined. In other words the unit intervals on the new axes must each vary inversely as the other.

As an extension of the idea of similarity for triangles, we may say that any two polygons which have their corresponding sides parallel and in proportion are similar. It follows that if any two corresponding lines are drawn in the polygons, these lines must be parallel.

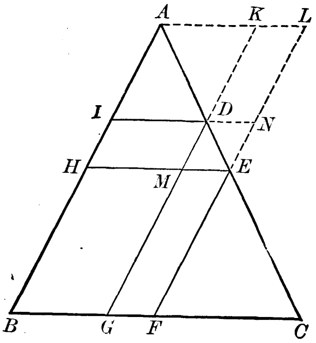

FIGURE 6.

XI. If on two sides of a triangle similar parallelograms be constructed, and on the third side a parallelogram with diagonals parallel to the diagonals of the other parallelograms, the area of this parallelogram will be equal to the difference of the areas of the other two.

The areas (Figure 7) of the parallelograms on *AB, CA, BC* are respectively four times the areas of the triangles *ABF, CAE, BCD*. If we take the unit parallelogram with sides parallel to the diagonals, it will suffice to prove that

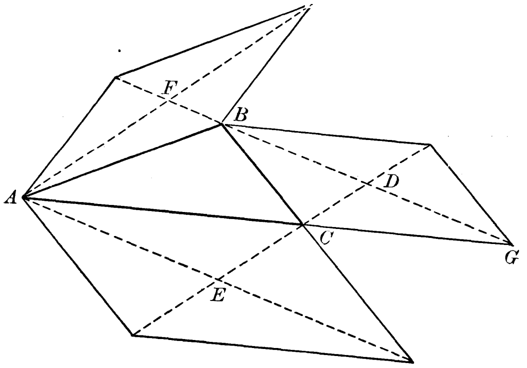

FIGURE 7.

$$FB \times AF = AE \times EC - BD \times CD,$$

for each of these areas is twice the area of the corresponding triangle. In the similar triangles *ACE* and *GCD*,

$$EC : CD : : AE : DG.$$

But by III, BD is equal to DG. And writing $AE = FB + BD$, we have

$$EC \times BD = CD \times FB + CD \times BD.$$

Add to each side the product $FB \times EC$. Then

$$EC(BD + FB) = CD \times BD + FB(CD + EC).$$

Hence

$$EC \times AE - CD \times BD = FB \times AF.$$

Non-Euclidean Rotation.

9. The group of parallel geometries determined by Postulates 1°–9°, which, notwithstanding its generality, gives rise, as we have seen, to some interesting and important theorems, may be subdivided by adding a set of postulates belonging to a second transformation which by analogy may be called rotation. It is this set of postulates which will differentiate our non-Euclidean geometry from the Euclidean.

The difference between our non-Euclidean rotation and the ordinary kind is that in addition to a fixed point, two *real* lines through the point remain unchanged. We may postulate for rotation:

10°. Any one point and only that one remains fixed.

This point may be called the center of rotation.

11°. Two lines through this point remain unchanged.

These lines may be called the fixed lines of the rotation.

12°. Any half-line (or ray) from the center, and lying in one of the angles determined by the fixed lines, may be turned into any other ray in the same angle, and this uniquely determines the rotation.

13°. The succession of two rotations about the same point is a rotation.

14°. The result of a rotation about O and a translation from O to O' is independent of the order in which the rotation and translation are carried out.

It follows immediately from 14° that the fixed lines in a rotation about any point O are parallel to the fixed lines in a rotation about any other point O'. All lines in the plane may now be divided into classes in such manner that neither translation nor rotation can change the classification. Namely,

(α) lines parallel to one of the fixed directions,

(β) lines parallel to the other of the fixed directions,

(γ) lines which lie in one of the pairs of vertical angles determined by the fixed directions,

(δ) lines which lie in the other pair of vertical angles determined by the fixed directions.

The lines of fixed direction, namely, the (α)-lines and (β)-lines, will be called *singular* lines.

A system of measurement may be set up for angles between rays[10] which issue from a point into one of the angles determined by the fixed lines through the point. For a succession of rotations may be used (in the same manner as the succession of translations was used to establish the measure of interval along a line). Thus if a line a is carried into a line a' and at the same time the line a' is carried into the line a'', the angles between a and a' and between a' and a'' are congruent and the measures of the angles are said to be equal. Now as the rotation may be repeated any number of times without reaching the fixed line, it is possible to find an angle $aa^{(n)}$ which shall be n times the angle aa'. We shall assume the postulate, analogous to the Archimedean:

15°. If a sufficient number of equal angles be laid off about a point from any initial ray, any ray of that class may be surpassed.

It thus appears that the angles between any given line and other lines of the same class may be placed into one-to-one correspondence with all positive and negative real numbers, just as the intervals from a point on a line may be thus correlated.[11] This constitutes a very great difference between our geometry and the Euclidean.

It is impossible to show from the preceding statements that any given figure maintains a constant area during rotation.[12] We shall therefore lay down the additional postulate:

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[10] The relations of order of all lines of a given class, (γ) or (δ), are the same as those of points on a line, as in 4°.

[11] The angle between two singular lines (α) and (β) can obviously not be measured. Such an angle, and also the angle between any line and a line of fixed direction, must be regarded as infinite.

[12] This matter may readily be discussed analytically. As axes of reference choose the fixed lines, and let u, v denote coordinates. As rotation is a linear transformation, the point P (u, v) and the transformed point P' (u', v') are connected by the equations

$$u' = au + bv + c, \qquad v' = du + ev + f.$$

As the lines $u = 0$ and $v = 0$ are fixed, these equations reduce to $u' = au$, $v' = ev$; and as rotation depends on only one parameter, we may write $e = \phi(a)$. The succession of two rotations is then expressed by

$$\begin{cases} u' = au \\ v' = \phi(a)v, \end{cases} \qquad \begin{cases} u'' = bu' \\ v'' = \phi(b)v', \end{cases} \qquad \begin{cases} u'' = abu \\ v'' = \phi(a)\phi(b)v, \end{cases}$$

16°. In rotation an area becomes an equal area.[13]

10. We are now prepared to discuss in some detail the general characteristics of our rotation. Consider (Figure 8) a series of rotations about O, whereby the point P assumes the positions P', P'',.... Let the parallelograms on OP, OP', OP'',.... as diagonals and with sides along the fixed lines be constructed. Then by 16° the areas of these parallelograms are equal, and in terms of the intervals on the fixed lines

$$OA \times OB = OA' \times OB'$$
$$= OA'' \times OB''.$$

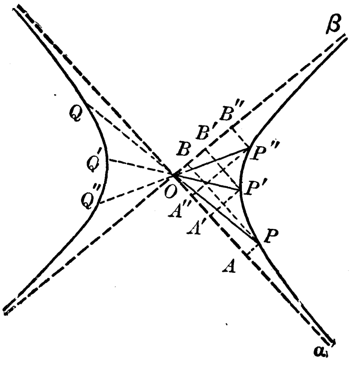

FIGURE 8.

The point P thus traces a curve which in ordinary geometry would be

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with the condition

$$\phi(a)\phi(b) = \phi(ab)$$

necessitated by 13°. This is a functional equation of which the only (continuous) solution is $\phi(a) = a^r$. Hence rotation must be of the form

$$u' = au, \ v' = a^r v.$$

The unit parallelogram on the axes of u and v is hereby transformed into a parallelogram on these same axes with intervals a and a^r along u and v. By VI the area of the new parallelogram is therefore a^{r+1}. If this is to be unity, $r = -1$. The transformation equations for rotation are therefore

$$u' = au, \quad v' = v/a,$$

where a is necessarily positive because points do not change from one side of the axes to another.

The intrinsic significance of these equations should not be overlooked. A rotation may be represented as a multiplication of all intervals along one of the fixed lines by a constant factor and a division of all intervals along the other fixed line by the same factor. Or, increasing the unit interval along one fixed line and decreasing it in the same ratio along the other is equivalent to a rotation. (This process effected along any other axes than the fixed lines would leave the area unchanged, but would not be a rotation). As the unit interval along one fixed line cannot be compared either by translation or by rotation with the unit along the other, and as one of these units is arbitrary, we have additional evidence that there is no natural zero of angle.

[13] Such a postulate is unnecessary in Euclidean geometry owing to the periodic nature of the Euclidean rotation. Postulate 16° could be replaced by one involving only the notion of symmetry between rotations in opposite directions.

considered a branch of a hyperbola.[14] Since, however, this curve is here generated by the rotation of a line OP about its terminus Q, we shall call this locus (taken with the other branch $Q\ Q'\ Q''$ symmetrically situated with respect to O) the *pseudo-circle.*

By means of such a rotation we are able to compare intervals upon any line with intervals upon any other line of the same class. For the intervals of the congruent radii OP, OP', OP'' will be called equal.

When we consider the fixed lines we observe that the effect of rotation is to carry the segment OA into OA' or OA''. It is therefore evident that segments are congruent by rotation which are incongruent by translation. This source of ambiguity exists only in the case of singular lines, for in no other case is it possible to compare two segments both by rotation and by translation. We may remove this ambiguity at once by stating that intervals along singular lines, although metrically comparable with intervals on other singular lines of the same class by translation, are all of zero magnitude when compared with intervals on any non-singular line. This will become more evident later.

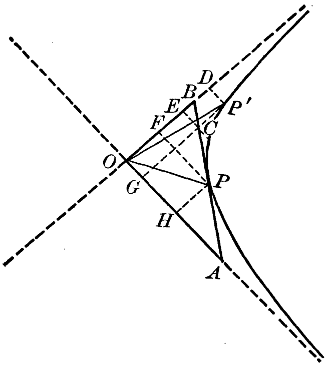

FIGURE 9.

Consider next (Figure 9) the intercept AB terminating on the fixed lines corresponding to a rotation with center at O. Let P be the middle point of the line, and C any other point. Through C draw a line parallel to OB, and on this line mark the point P' such that the area $ODP'G$ equals the area $OFPH$. The area $OECG$ is less than each of these by X. Hence P' lies on the further side of AB from O. But P' is a point on the pseudo-circle through P concentric with O, as we have just seen. Since C was any point of AB, it follows that P' may be any point of the pseudo-circle. Hence as the line AB meets the pseudo-circle at P and only at P, it is tangent to the curve. As a species of converse, we may state the theorem:

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[14] There is no special significance in the fact that a rectangular hyperbola is drawn in the figure and that the fixed lines α, β are perpendicular in the Euclidean sense; in subsequent figures the singular lines are often oblique. From the non-Euclidean viewpoint the question of perpendicularity or obliquity of the singular lines is of course meaningless.

XII. The tangent to a pseudo-circle lies between the curve and its center, and the portion of the tangent intercepted between the two fixed lines is bisected at the point of tangency.

11. In a pseudo-circle the radius and the tangent at its extremity are said to be perpendicular. Or in virtue of XII we may say that the perpendicular from any point O to any non-singular line is the line from O to the middle point of that segment of the line which is intercepted by the fixed lines through O. The construction of a perpendicular to any line of class (γ) or (δ) at a point of the line is equally simple.

By the aid of propositions concerning similar triangles, the following theorems concerning perpendiculars are readily proved.

XIII. If a line a is perpendicular to a line b, then b is perpendicular to a.

XIV. Through any point one and only one perpendicular can be drawn to any line.

XV. All lines perpendicular to the same line are parallel.

XVI. The singular line of one class which is drawn through the intersection of any two perpendicular lines will bisect the segment intercepted by these lines upon any singular line of the other class (Figure 10).[15]

XVII. The perpendicular to a (γ)-line is a (δ)-line, and vice versa.

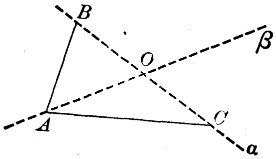

FIGURE 10.

Intervals along lines of class (δ) cannot be compared by congruence with intervals along lines of the (γ) class. We may, therefore, arbitrarily define equality of intervals between the two classes. *If two mutually perpendicular lines are drawn from any point and terminate on a singular line, the intervals of these lines will be said to be equal.*[16] The consistency of this definition is readily proved.

The definition of perpendicularity is such that if two lines are perpendicular they must remain perpendicular after a translation or rotation. The former case is obvious, and the latter becomes so when the lines are considered as radius and tangent in a pseudo-circle generated by the rotation; the more general case in which neither of the perpendicular lines passes through the center of rotation then follows with the aid of XV. It is important to observe one peculiar characteristic of our rotation, namely that two perpendicular lines approach each other and the fixed line between them scissor-wise, as may be seen, in Figure 11, where OC and OD become respectively OC' and OD', OC'' and OD'', $\cdots$ The pseudo-circles traced by OC and OD may be called conjugate pseudo-circles, since the interval OC equals the interval OD, the lines CD, $C'D'$, $\cdots$, being singular, and bisected by a fixed line.

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[15] In the figure BO and OC are equal, and AB and AC are perpendicular.

[16] In Figure 10, the intervals AC and AB are therefore equal by this definition.

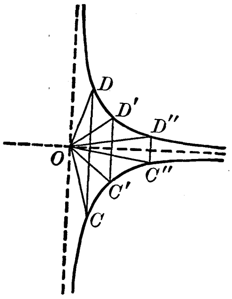

FIGURE 11.

Since two mutually perpendicular lines approach, during rotation about their point of intersection, the same fixed line, we may extend our definition of perpendicularity by regarding every singular line as perpendicular to itself. This extension is also suggested by the fact that the fixed line may be considered an asymptote of a pseudo-circle. Special caution must be given against the idea that a singular line of one class is perpendicular to a singular line in the other class. The peculiarities of singular lines will become clearer in the work on vector analysis.

12. A triangle of which two sides are perpendicular will be called a right triangle, and the third side will be called the hypotenuse. A parallelogram of which the two adjacent sides are perpendicular and of equal interval will be called a square. The following theorem is obvious:

XVIII. One diagonal of every square is a singular line and the other diagonal is a singular line of the other class.

XIX. *Pythagorean Theorem.* The area of the square on the hypotenuse of a right triangle is equal to the difference of the areas of the squares on the other two sides.

For by XVIII the diagonals of the squares are lines of fixed direction, and hence parallel each to each. The squares on the two legs are similar. And the proposition is evidently a special case of XI. (In Figure 7 if the dotted lines are singular lines, the lines AC and BC are so drawn as to be approximately perpendicular.)

XX. Any two squares whose sides are of unit interval are equal in area.

For by suitable translation and rotation one may be brought into coincidence with the other. The unit of area will henceforth be taken as the area of a square whose sides are of unit interval. Hence follows:

Cor. The area of any rectangle is the product of the intervals of two adjoining sides.

We may therefore obtain from XIX the theorem

XXI. The square of the interval of the hypotenuse of a right triangle is equal to the difference in the squares of the intervals of the other two sides.

Cor. The perpendicular from a point to a line has a *greater* interval than any other line of the same class drawn from the given point to the given line.

Having now given a final definition of the measure of area, we may define the unit of angle. The radius of the pseudo-circle, in advancing by rotation over equal angles, necessarily sweeps out equal areas (by 16°). Hence by the familiar argument sectorial areas in any pseudo-circle are proportional to the angles at the center. The unit angle will be taken as that angle which, in a pseudo-circle of unit radius, encloses a sectorial area of one-half the unit area.

Vectors and Vector Algebra.

13. Translation or the parallel-transformation leads at once to the consideration of vectors. We have shown that when a translation carries A into B and A' into B' the directed segments AB and $A'B'$ are parallel and congruent (Cor. to II). Hence a translation may be represented by a vector, that is, by any directed segment laid of from any origin and having the same interval and direction as AB. The succession of two translations is represented by the sum of their corresponding vectors. The addition and subtraction of vectors and their multiplication by scalars follows the usual laws (by §§ 5–7).

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are laid off from a common origin, the parallelogram constructed on the vectors is called their outer product $\mathbf{a}\times\mathbf{b}$, and the magnitude of this product will be taken numerically equal to the area of the parallelogram.[17] We must bear in mind that not this magnitude (nor yet a vector perpendicular to the plane), but the parallelogram itself is the outer product. We may, however, represent the outer product by any other closed figure of equal area, provided that it is taken with the same sign. The sign attributed to an area does not arise from any positive or negative geometric characteristics of the area itself, but from an interpretation or convention concerning the way in which one area is considered as generated relative to another, and is required for analytic work. We shall make the convention that $\mathbf{a}\times\mathbf{b}$ and $(-\mathbf{a})\times\mathbf{b}$ or $\mathbf{a}\times(-\mathbf{b})$ have opposite signs.

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[17] Our vector notation will be based upon that of Gibbs, and is identical with that employed by Lewis (Four dimensional Vector Analysis, These Proceedings, **46**, 163–181) except in the designation of the inner product which we shall define as in that paper, but represent by $\mathbf{a}\cdot\mathbf{b}$ instead of $\mathbf{ab}$; the latter form will be reserved to denote the dyad. The scalar magnitude of a vector will be represented by the same letter in italic type.

The outer product of a vector by itself or by any parallel vector is zero, because the parallelogram determined by these vectors has zero area; thus $\mathbf{a}\times\mathbf{a} = 0$. The associative law for a scalar factor is valid, because multiplying one side of a parallelogram by a number multiplies the area by that number; thus

$$(n\mathbf{a})\times\mathbf{b} = n\,\mathbf{a}\times\mathbf{b} = \mathbf{a}\times(n\mathbf{b}).$$

The distributive laws,

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}, \qquad (\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c},$$

also hold; for inspection shows that the parallelogram $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$ is equal to $\mathbf{a}\times\mathbf{b}$ plus $\mathbf{a}\times\mathbf{c}$. The anti-commutative law,

$$\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a},$$

holds; for

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = \mathbf{a}\times\mathbf{a} + \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{b} = 0.$$

Hence

$$\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}.$$

14. Thus far we have proceeded by means of the parallel-transformation alone. It is evident that this much of vector algebra is common to all geometries, including the Euclidean and our non-Euclidean geometry, in which there is such a parallel-transformation. The other type of product, the inner product, cannot be defined without some concept of rotation or perpendicularity, or its equivalent.

We shall so define this inner product $\mathbf{a}\cdot\mathbf{b}$ that it obeys the associative law for a scalar factor and the distributive and commutative laws, namely,

$$(n\mathbf{a})\cdot\mathbf{b} = n\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(n\mathbf{b}),$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c},$$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a},$$

and furthermore remains invariant during rotation.

As the fixed lines are fundamental in rotation it is sometimes expedient to resolve vectors into components along these directions. Let $\mathbf{p}$ and $\mathbf{q}$ be definite vectors in the two fixed lines; any vector in the plane may be written as $\mathbf{r} = x\mathbf{p} + y\mathbf{q}$. By the postulated formal laws,

$$\mathbf{r}\cdot\mathbf{r} = x^2\,\mathbf{p}\cdot\mathbf{p} + y^2\,\mathbf{q}\cdot\mathbf{q} + 2xy\,\mathbf{p}\cdot\mathbf{q}.$$

We may now note that by rotation a vector along a fixed line is converted into a multiple of that vector. If $\mathbf{p}$ becomes $n\mathbf{p}$, and the inner product $\mathbf{p}\cdot\mathbf{p}$ remains invariant, then $\mathbf{p}\cdot\mathbf{p} = n^2\,\mathbf{p}\cdot\mathbf{p}$; whence it is obvious that $\mathbf{p}\cdot\mathbf{p} = 0$. In general: The inner product of any singular vector by itself is zero, and this suffices to characterize a singular vector. Hence $\mathbf{r}\cdot\mathbf{r}$ reduces to

$$\mathbf{r}\cdot\mathbf{r} = 2xy\,\mathbf{p}\cdot\mathbf{q}.$$

Before proceeding further with the definition of the inner product, we may observe that the signs of x and y are determined by that one of the four angles (made by the fixed lines) in which $\mathbf{r}$ lies. According, then, as x and y have the same sign or different signs, the vector $\mathbf{r}$ belongs to one or the other of the classes (γ) or (δ), and the product $\mathbf{r}\cdot\mathbf{r}$ will have one sign or the other. These considerations suffice to show that if $\mathbf{r}$ and $\mathbf{r}'$ are two vectors, and if $\mathbf{r}\cdot\mathbf{r}$ and $\mathbf{r}'\cdot\mathbf{r}'$ have the same sign, the vectors are of the same class, but if $\mathbf{r}\cdot\mathbf{r}$ and $\mathbf{r}'\cdot\mathbf{r}'$ are of opposite sign, $\mathbf{r}$ and $\mathbf{r}'$ are of different classes. We have here a marked departure from Euclidean geometry, in which the inner product of a real vector by itself is always positive.

We are now in a position to complete the definition of the inner product by stating that the product is a scalar, and that the product of a vector by itself is equal to the square of the interval of the vector, taken positively if the vector is of class (γ), negatively if of class (δ). This does not imply any dissymmetry between the classes (γ) and (δ), but is only such a convention as is often made with respect to sign.

The equation $\mathbf{r}\cdot\mathbf{r} = 2xy\,\mathbf{p}\cdot\mathbf{q}$ shows that the inner product of any singular vector and any singular vector of the other class is equal to one-half the inner product by itself of the diagonal of their parallelogram.

The inner product of any vector and a perpendicular vector is zero.[17] For by XVI it is evident that if $\mathbf{p}$ and $\mathbf{q}$ be the components along the fixed directions of any vector $\mathbf{r}$, so that $\mathbf{r} = \mathbf{p} + \mathbf{q}$, then $\mathbf{p} - \mathbf{q}$ is a perpendicular vector, and in general any perpendicular vector $\mathbf{r}'$ has the form $n(\mathbf{p} - \mathbf{q})$. Hence

$$\mathbf{r}'\cdot\mathbf{r} = n(\mathbf{p} - \mathbf{q})\cdot(\mathbf{p} + \mathbf{q}) = n(\mathbf{p}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{q} - \mathbf{q}\cdot\mathbf{p} - \mathbf{q}\cdot\mathbf{q}) = 0.$$

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[17] The fact that the inner product of a singular vector by itself vanishes justifies our convention that a singular line is perpendicular to itself.

The inner product of any two vectors is equal to the inner product of either one by the projection of the other along it. For either vector may be resolved into two vectors one of which is parallel and the other perpendicular to the other vector. Thus $\mathbf{b}$ may be written as $n\mathbf{a} + \mathbf{a}'$, where $n\mathbf{a}$ is the projection of $\mathbf{b}$ on $\mathbf{a}$, and $\mathbf{a}'$ is perpendicular to $\mathbf{a}$. Therefore

$$\mathbf{b}\cdot\mathbf{a} = n\mathbf{a}\cdot\mathbf{a} + \mathbf{a}'\cdot\mathbf{a} = n\mathbf{a}\cdot\mathbf{a},$$

which was to be proved. Geometrically the only puzzling case is that in which the vectors are of different classes. Let OA (Figure 12) be a vector of class (γ) and OB of class (δ). The projections of OA on OB and of OB on OA are respectively OB' and OA'. Note that whereas OB' extends in the same direction as OB, the vector OA' extends along the opposite direction to OA. Thus OB' is a positive multiple of OB, whereas OA' is a negative multiple of OA. But the inner product of OB by itself is negative, since the vector is of class (δ), while the inner product of OA by itself is positive, since the vector is of class (γ). Hence the inner product of OA and OB has the same sign, whichever way the projection is taken.

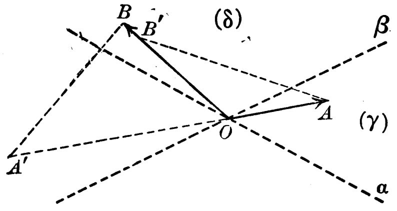

FIGURE 12.

In obtaining the inner product of a singular and a non-singular vector by projecting one upon the other, it is necessary to project the singular vector upon the non-singular vector; for it is impossible to make a perpendicular projection upon a singular vector. In case both vectors are singular the method of perpendicular projection fails entirely, and we must use analytical methods (or have recourse to parallel projection).

15. It will often be convenient to select two mutually perpendicular lines as axes of reference. We will denote[18] by $\mathbf{k}_1$ and $\mathbf{k}_4$ unit vectors along such axes, $\mathbf{k}_1$ being the vector of the (γ)-class, and $\mathbf{k}_4$ of class (δ). For these vectors we have the rules of multiplication

$$\mathbf{k}_1\cdot\mathbf{k}_1 = 1, \qquad \mathbf{k}_4\cdot\mathbf{k}_4 = -1, \qquad \mathbf{k}_1\cdot\mathbf{k}_4 = \mathbf{k}_4\cdot\mathbf{k}_1 = 0.$$

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[18] We reserve the symbols $\mathbf{k}_2$ and $\mathbf{k}_3$ for other unit vectors of class (γ) in space of higher dimensions.

Any two vectors **a** and **b** may be written in the form

$$\mathbf{a} = a_1\mathbf{k}_1 + a_4\mathbf{k}_4, \qquad \mathbf{b} = b_1\mathbf{k}_1 + b_4\mathbf{k}_4,$$

and the inner product is then, by the distributive law,

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 - a_4b_4.$$

In terms of these unit vectors we may also express outer products. If we write, for brevity, $\mathbf{k}_{14} = \mathbf{k}_1\times\mathbf{k}_4$, the rules for outer multiplication are

$$\mathbf{k}_{14} = -\mathbf{k}_{41}, \qquad \mathbf{k}_{11} = \mathbf{k}_{44} = 0.$$

The outer product of the vectors **a** and **b** is therefore

$$\mathbf{a}\times\mathbf{b} = (a_1b_4 - a_4b_1)\,\mathbf{k}_{14}.$$

Since $\mathbf{k}_{14}$ represents a parallelogram of unit area, the question arises as to why we write $\mathbf{k}_1\times\mathbf{k}_4$ as $\mathbf{k}_{14}$ and not simply $\mathbf{k}_1\times\mathbf{k}_4 = 1$. The answer is that the outer product $\mathbf{a}\times\mathbf{b}$ possesses a certain dimensionality, which, it is true, is not exhibited in a marked degree until we proceed into a space of higher dimensions, but which renders it undesirable to regard the outer product as merely a scalar. We may call it a pseudo-scalar, and later extend this designation to n-dimensional figures in a manifold of n dimensions.

Every vector in two dimensional space uniquely determines, except for sign, another vector, namely, the one equal in interval and perpendicular to the first. This vector will be called the complement of the given vector. To specify this sign, the complement $\mathbf{a}^*$ of the vector **a** may be defined as the inner product of **a** and the unit pseudo-scalar $\mathbf{k}_{14}$, namely, $\mathbf{a}^* = \mathbf{a}\cdot\mathbf{k}_{14}$, where the laws of this inner product are

$$\mathbf{k}_1\cdot\mathbf{k}_{14} = -\mathbf{k}_4, \qquad \mathbf{k}_4\cdot\mathbf{k}_{14} = -\mathbf{k}_1.$$

Thus if $\mathbf{a} = a_1\mathbf{k}_1 + a_4\mathbf{k}_4$, then for the complement

$$\mathbf{a}^* = (a_1\mathbf{k}_1 + a_4\mathbf{k}_4)^* = (a_1\mathbf{k}_1 + a_4\mathbf{k}_4)\cdot\mathbf{k}_{14} = -a_4\mathbf{k}_1 - a_1\mathbf{k}_4.$$

This type of multiplication, as will be seen later, obeys all the general laws of inner products (§§ 27, 29).

Referred to a set of perpendicular unit vectors, the singular vectors take the form $n(\pm\mathbf{k}_1 \pm \mathbf{k}_4)$. The complement of a singular vector is

$$n(\pm\mathbf{k}_1 \pm \mathbf{k}_4)^* = n(\pm\mathbf{k}_1 \pm \mathbf{k}_4)\cdot\mathbf{k}_{14} = n(\mp\mathbf{k}_4 \mp \mathbf{k}_1),$$

that is, the complement of a singular vector is its own negative.

We may extend the idea of complements to scalars and pseudo-scalars. The complement of the scalar n will be defined as the pseudo-scalar $n\mathbf{k}_{14}$; the complement of the pseudo-scalar $n\mathbf{k}_{14}$ will be defined as the scalar $-n$. This may be written

$$(n\mathbf{k}_{14})^* = n\mathbf{k}_{14}\cdot\mathbf{k}_{14} = -n,$$

thus establishing the convention $\mathbf{k}_{14}\cdot\mathbf{k}_{14} = -1$. It may readily be shown that, for any two singular vectors $\mathbf{p}$ and $\mathbf{q}$ of different class, the outer product is the complement of the inner product, that is,

$$\mathbf{p}\times\mathbf{q} = (\mathbf{p}\cdot\mathbf{q})\mathbf{k}_{14}.$$

In other words the inner and outer products of singular vectors are numerically equal.

Some Differential Relations.

16. As the inner product $\mathbf{r}\cdot\mathbf{r}$ of a vector by itself is numerically equal to the square of the interval of the vector $\mathbf{r}$, the equation of the unit pseudo-circle of which the radii are all (γ)-lines is $\mathbf{r}\cdot\mathbf{r} = 1$; and the equation of the conjugate unit pseudo-circle of which the radii are (δ)-lines is $\mathbf{r}\cdot\mathbf{r} = -1$. As the tangents to a pseudo-circle are perpendicular to the radii, they must be of opposite class. A pseudo-circle of which any tangent is a (δ)-line (the radii being (γ)-lines) is called a (δ)-pseudo-circle; and a pseudo-circle of which any tangent is a (γ)-line (the radii being (δ)-lines) is called a (γ)-pseudo-circle. In general if a curve has tangents which are all of the same class (δ) or (γ), the curve may be designated as a (δ)- or a (γ)-curve; the normals to the curve will then be respectively of the opposite class (γ) or (δ). The interval of the arc of any such curve will be the limit of the sum of the intervals of the infinitesimal chords along the arc. We shall not be obliged to consider any curve which is not altogether of one class as here defined.

As $d\mathbf{r}$ is the infinitesimal chord as a vector quantity, the formula for the scalar arc is

$$s = \int \sqrt{d\mathbf{r}\cdot d\mathbf{r}} \qquad \text{or} \qquad s = \int \sqrt{-\,d\mathbf{r}\cdot d\mathbf{r}}$$

according as the curve is a (γ)- or a (δ)-curve.

The sectorial area in a unit pseudo-circle may be regarded as the sum of infinitesimal right triangles, of which the area is numerically equal to $\frac{1}{2}\mathbf{r}\times d\mathbf{r}$ if $\mathbf{r}$ is drawn from the center. The numerical value of the area is therefore one-half the numerical value of $d\mathbf{r}$, that is, one-half the infinitesimal interval of arc. From our definition of unit angle (§ 12), it is evident that an angle is equal to the arc subtended upon a unit pseudo-circle centered at the vertex of the angle. This might, in fact, have been made the definition of the measure of angle. It is evident from these considerations that a rotation turns all non-singular lines through the same angle.

Angles may be classified according to the classes of their sides. If the two sides are (γ)-lines, the angle will be designated as of class $(\gamma\gamma)$; if they are (δ)-lines, the angle is of class $(\delta\delta)$. Consideration of angles $(\gamma\delta)$, which have one side a (γ)-line and the other a (δ)-line, and which cannot be generated by rotation, need not detain us here. (See Appendix.)

If any line (Figure 13) through the center be taken from which to measure angle, position upon the unit pseudo-circle may be expressed parametrically in terms of the angle as follows. Let the given line be a line of class (γ) (the pseudo-circle then being of class (δ)), and construct the perpendicular line of class (δ). These two lines may be taken respectively as axes of x_1 and x_4 with the unit vectors $\mathbf{k}_1$ and $\mathbf{k}_4$ along them. The equation of the unit pseudo-circle is then

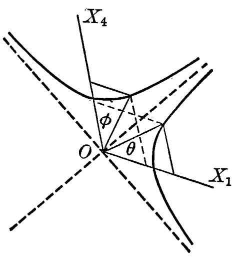

FIGURE 13.

$$\mathbf{r}\cdot\mathbf{r} = (x_1\mathbf{k}_1 + x_4\mathbf{k}_4)\cdot(x_1\mathbf{k}_1 + x_4\mathbf{k}_4) = x_1^2 - x_4^2 = 1.$$

The differential of angle or arc is in this case

$$d\theta = ds = \sqrt{-d\mathbf{r}\cdot d\mathbf{r}} = \sqrt{(\mathbf{k}_1dx_1+\mathbf{k}_4dx_4)\cdot(\mathbf{k}_1dx_1+\mathbf{k}_4dx_4)} = \sqrt{dx_4^2-dx_1^2}$$

Whence, by differentiation of $x_1^2 - x_4^2 = 1$,

$$\int d\theta = \int ds = \int \frac{dx_4}{\sqrt{1 + x_4^2}} = \int \frac{dx_1}{\sqrt{x_1^2 - 1}},$$

and $\qquad x_1 = \cosh\theta, \qquad x_4 = \sinh\theta,$

where θ is the angle between the x_1-axis and the radius vector, and therefore of the class $(\gamma\gamma)$. If the given line had been of class (δ) (the pseudo-circle of class (γ)), and if the angle ϕ had been of class $(\delta\delta)$ measured from the x_4-axis to the radius vector, the results would have been

$$x_1 = \sinh\phi, \qquad x_4 = \cosh\phi,$$

with $x_1^2 - x_4^2 = -1$ as the equation of the pseudo-circle.

If now in general **r** be the radius of any pseudo-circle, the foregoing results may readily be generalized, and we obtain the following pair of equations.

$$\begin{aligned} x_1 &= r\cosh\theta, & x_4 &= r\sinh\theta, & x_4 &= x_1\tanh\theta; \\ x_1 &= r\sinh\phi, & x_4 &= r\cosh\phi, & x_1 &= x_4\tanh\phi. \end{aligned} \tag{1}$$

In the first case **r** is a (γ)-vector and θ is a $(\gamma\gamma)$-angle; in the second, **r** is a (δ)-vector and ϕ is a $(\delta\delta)$-angle. We thus have equations which express the relations between the hypotenuse and the sides of any right triangle in terms of one angle. The inclination of the vector **r** to the axes $\mathbf{k}_1$ or $\mathbf{k}_4$ in the respective cases is the angle

$$\theta = \tanh^{-1}\frac{x_4}{x_1} \qquad \text{or} \qquad \phi = \tanh^{-1}\frac{x_1}{x_4};$$

and the slope of **r** relative to the axes is the hyperbolic tangent of the angle, not the trigonometric tangent.

17. Consider next any curve of class (δ). Let

$$s = \int \sqrt{dx_4^2 - dx_1^2}$$

denote scalar arc along the curve, and let **r** be the radius vector from a fixed origin to any point of the curve. Then the derivative

$$\mathbf{w} = \frac{d\mathbf{r}}{ds} = \mathbf{k}_1\frac{dx_1}{ds} + \mathbf{k}_4\frac{dx_4}{ds} \tag{2}$$

is a unit vector tangent to the curve. If this vector makes the angle ϕ with the axis $\mathbf{k}_4$, so that the slope of the curve is

$$v = \tanh\phi = \frac{dx_1}{dx_4}, \tag{3}$$

the components of the vector are

$$\frac{dx_1}{ds} = \sinh\phi = \frac{v}{\sqrt{1-v^2}}, \qquad \frac{dx_4}{ds} = \cosh\phi = \frac{1}{\sqrt{1-v^2}}, \tag{4}$$

and

$$\mathbf{w} = \frac{1}{\sqrt{1-v^2}}(v\mathbf{k}_1 + \mathbf{k}_4). \tag{5}$$

If we had chosen a different set of perpendicular axes $\mathbf{k}_1'$, $\mathbf{k}_4'$, where $\mathbf{k}_4'$ makes an angle $\psi = \tanh^{-1} u$ with $\mathbf{k}_4$, so that the inclination of $\mathbf{w}$ to $\mathbf{k}_4'$ is $\phi' = \phi - \psi$, the new components of $\mathbf{w}$ would be

$$\frac{dx_1'}{ds} = \sinh\phi' = \cosh\phi\cosh\psi - \sinh\phi\sinh\psi = \frac{v'}{\sqrt{1-v'^2}}$$

$$= \frac{v-u}{\sqrt{1-v^2}\sqrt{1-u^2}},$$

$$\frac{dx_4'}{ds} = \cosh\psi' = \cosh\phi\cosh\psi - \sinh\phi\sinh\psi = \frac{1}{\sqrt{1-v'^2}}$$

$$= \frac{1-vu}{\sqrt{1-v^2}\sqrt{1-u^2}},$$

where

$$v' = \frac{dx_1'}{dx_4'} = \tanh\phi' = \frac{\tanh\phi - \tanh\psi}{1-\tanh\phi\tanh\psi} = \frac{v-u}{1-vu}. \tag{6}$$

It will be convenient to have a general equation for the components of a vector upon one set of axes in terms of its components on another set. Let $\mathbf{k}_1$, $\mathbf{k}_4$ be one set of perpendicular unit vectors, and $\mathbf{k}_1'$, $\mathbf{k}_4'$ another set. If the angle from $\mathbf{k}_1$ to $\mathbf{k}_1'$ be ψ, the angle from $\mathbf{k}_4$ to $\mathbf{k}_4'$ is also ψ by § 16. The products

$$\mathbf{k}_1\cdot\mathbf{k}_1' = \cosh\psi, \qquad \mathbf{k}_4\cdot\mathbf{k}_4' = -\cosh\psi,$$

$$\mathbf{k}_1\cdot\mathbf{k}_4' = \sinh\psi, \qquad \mathbf{k}_1'\cdot\mathbf{k}_4 = -\sinh\psi,$$

follow from (1). To obtain the transformation equations we write

$$\mathbf{r} = x_1\mathbf{k}_1 + x_4\mathbf{k}_4 = x_1'\mathbf{k}_1' + x_4'\mathbf{k}_4',$$

and multiply by $\mathbf{k}_1$, $\mathbf{k}_4$, $\mathbf{k}_1'$, $\mathbf{k}_4'$;

$$\begin{aligned} \mathbf{r}\cdot\mathbf{k}_1 &= x_1 = x_1'\cosh\psi + x_4'\sinh\psi, \\ -\mathbf{r}\cdot\mathbf{k}_4 &= x_4 = x_1'\sinh\psi + x_4'\cosh\psi, \\ \mathbf{r}\cdot\mathbf{k}_1' &= x_1' = x_1\cosh\psi - x_4\sinh\psi, \\ -\mathbf{r}\cdot\mathbf{k}_4' &= x_4' = -x_1\sinh\psi + x_4\cosh\psi. \end{aligned} \tag{7}$$

Curvature in our non-Euclidean geometry is defined, as is ordinary geometry, as the rate of turning of the tangent relative to the arc. As $\mathbf{w}$ is a unit tangent, $d\mathbf{w}$ is perpendicular to $\mathbf{w}$ and in magnitude is equal to the differential angle through which $\mathbf{w}$ turns. Hence

$$\mathbf{c} = \frac{d\mathbf{w}}{ds} = \frac{d^2\mathbf{r}}{ds^2} \tag{8}$$

is the curvature, taken as a vector normal to the curve. Hence

$$\mathbf{c} = \left[\frac{\mathbf{k}_1}{(1-v^2)^2} + \frac{v\mathbf{k}_4}{(1-v^2)^2}\right]\frac{dv}{dx_4}. \tag{9}$$

In magnitude the curvature is

$$\sqrt{\mathbf{c}\cdot\mathbf{c}} = \frac{\dfrac{dv'}{dx_4}}{(1-v^2)^{\frac{3}{2}}} = \frac{\dfrac{d^2x_1}{dx_4{}^2}}{\left[1-\left(\dfrac{dx_1}{dx_4}\right)^2\right]^{\frac{3}{2}}}.$$

Relative to axes $\mathbf{k}_1'$, $\mathbf{k}_4'$, the result is

$$\mathbf{c} = \left[\frac{\mathbf{k}_1'}{(1-v'^2)^2} + \frac{v'\mathbf{k}_4'}{(1-v'^2)^2}\right]\frac{dv'}{dx_4'}$$

$$= \left[\frac{(1-uv)\,\mathbf{k}_1'}{(1-v^2)^2\sqrt{1-u^2}} + \frac{(v-u)\,\mathbf{k}_4'}{(1-v^2)^2\sqrt{1-u^2}}\right].$$

In complete analogy with the circle in Euclidean geometry the pseudo-circle in our non-Euclidean geometry has a curvature of constant magnitude throughout. The curvature of any other curve may always be represented as the curvature of the osculating pseudo-circle, and in magnitude is inversely proportional to the radius of that pseudo-circle.

Kinematics in a Single Straight Line.

18. Before proceeding to the discussion of the non-Euclidean geometry of more than two dimensions we may consider some simple but fundamental problems of physics which may be treated with the aid of the results which we have already obtained.

The science of kinematics involves a four dimensional manifold, of which three of the dimensions are those of space, and one that of time. By neglecting two of the spacial dimensions, in other words by restricting our considerations to the motion of a particle[19] in a single straight line, kinematics becomes merely a two dimensional science. The theorems of kinematics, not in the classical form, but in the form given to them by the principle of relativity, are simply theorems in our non-Euclidean geometry.

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[19] By particle we do not as yet mean a material particle but merely an identifiable point in motion.

The units of distance and time, namely the centimeter and second, were chosen without reference to each other. Retaining the centimeter as the unit of distance, we may take as the unit of time one which had been frequently suggested as the rational unit long before the principle of relativity was enunciated, namely, the second divided by 3×10^{10}, or the time required by light in free space to travel one centimeter. The velocity of light then becomes unity.

Let us consider in our geometry two perpendicular lines, and measure along the (γ)-line extension in space, along the (δ)-line extension in time. Then any point in the plane will represent a given position at a given time. We are considering the motion of a particle along a specified straight line in space. If x denotes distance along the line from a chosen origin, then in terms of our previous nomenclature, we shall take $x = x_1$ and $t = x_4$. The $\mathbf{k}_4$- or t-axis, or any line in the xt-plane parallel to this axis, represents the locus in time of a particle which does not change its position in space, in other words, of a stationary particle. Any straight line of the (δ)-class making a non-Euclidean angle ψ with $\mathbf{k}_4$, represents the locus in space and time of a particle moving with a constant velocity

$$u = \frac{dx}{dt} = \tanh \psi$$

A singular line in our plane represents a velocity $u = 1$, and is the locus of a particle moving with the velocity of light.

We have seen that in our plane no pair of perpendicular lines is better suited to serve as coordinate axes than any other pair. If then we consider (Figure 14) two (δ)-lines, marked t and t', and the respectively perpendicular (γ)-lines, marked x and x', and if we regard the first (δ)-line as the locus of a stationary particle and the second as the locus of a moving particle, we might expect to find that we could equally well regard the second (δ)-line as the locus of a particle at rest and the first as the locus of a moving particle. And this is, in fact, the first postulate of the principle of relativity. The one relation between the two lines, which is independent of any assumption as to which line is the locus of a stationary point, is

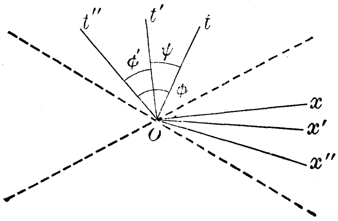

FIGURE 14.

the angle ψ whose hyperbolic tangent is the *relative* velocity which is the same by either of the assumptions.

If now we have a third (δ)-line t'' making an angle ϕ with the first (δ)-line, and ϕ' with the second, where $\phi' = \phi - \psi$, and if we call the relative velocities corresponding to these angles

$$v = \tanh \phi, \qquad v' = \tanh \phi', \qquad u = \tanh \psi,$$

then it is not true that $v' = v - u$, but since $\phi' = \phi - \psi$,

$$v' = \frac{v - u}{1 - vu}$$

by (6). This is the theorem regarding the addition of velocities obtained by Einstein.[20] The true significance of this result cannot be emphasized too strongly, namely, that the velocity as such can only be determined after a set of axes have been arbitrarily chosen; relative velocity, however, has a meaning independent of any coordinate system. Furthermore it is not the relative velocities, but the non-Euclidean angles, which are their hyperbolic anti-tangents, which are simply additive. If we were constructing a new system of kinematics uninfluenced by the historical development of the science, it might be preferable to make these angles fundamental rather than the velocities.

Suppose that from a given (δ)-line we lay off successively equal angles, so that each line determines with the preceding line the same relative velocity, then the angle measured from the given line increases without limit, but its hyperbolic tangent, which is the velocity relative to this line, approaches unity, that is, the velocity of light. The relative velocity, therefore, determined by any two (δ)-lines whatever, is less than the velocity of light. The velocity of light itself appears the same regardless of the choice of coordinate axes. This is the second postulate of the principle of relativity. Indeed if angle, instead of relative velocity, had been made fundamental, the motion of light, as compared with all other motions, would have been characterized by an infinite value of the angle.

19. Let us return to our figure and consider once more the lines that have been marked t, t', and x, x'. If we take the t-line as the locus of a stationary particle, then all points along the line x or along any parallel line are said to be simultaneous, for along any line perpendicular to the t-axis the value of t is constant. In like manner if we consider the t'-line as the locus of a particle at rest, then simultaneous points are those along x' or along lines parallel to x'. Hence points which are simultaneous from one point of view, are not simultaneous from the other. In fact any two points through which a line of class (γ) can be drawn may be regarded as simultaneous by choosing this (γ)-line as the axis x, and the perpendicular line as the axis t. Similarly any two points through which a (δ)-line can be drawn may be regarded as having the same spacial position; in other words any point may be taken as a point at rest.

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[20] Einstein, Jahrb. d. Radioak, **4,** 423.

It thus appears that the measurements of time and space are determined only relative to some selected set of axes. Further to exhibit this fact, and to determine the relations which exist between the measures of time and space when different sets of axes are chosen, let us consider (Figure 15) two parallel (δ)-lines in our non-Euclidean plane. These lines represent the loci of two particles which have no relative velocity. Let any set of axes of time and space be drawn. The constant intervals cut off by the two parallel (δ)-lines from the x-axis and all lines parallel to this axis represent the constant distance, as measured by these axes, between the two particles at any time. The constant intervals cut off by the two parallel (δ)-lines on the t-axis and all lines parallel thereto represent the constant interval of time as measured by these axes, which must elapse between the instant when one of the particles has a certain position (upon the line in which we are considering rectilinear motion as taking place) and the instant when the other of the particles has this same position.

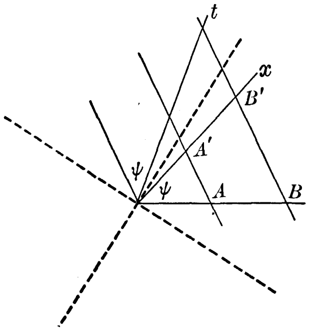

FIGURE 15.

One particular choice of axes is especially simple, namely, that in which the t-axis is parallel to the two (δ)-lines, and the x-axis is perpendicular. Relative to this assumption of axes the particles are at rest. The distance between them is AB. If another set of axes is drawn, the particles appear to be in motion, and the distance between them is taken as $A'\,B'$. If ψ denotes the angle between the axes, the projection of $A'B'$ on AB is equal to AB,

$$AB = A'B' \cosh \psi = \frac{A'B'}{\sqrt{1 - u^2}},$$

where u is the relative velocity determined by ψ. Or,

$$A'B' = AB \operatorname{sech} \psi = AB \sqrt{1-u^2}.$$

That is to say, the distance $A'B'$ between the particles when considered in motion with the velocity u is to the distance AB between the particles when considered at rest as $\sqrt{1-u^2} : 1$. This statement embodies Lorentz's theory of the shortening of distances in the direction of motion.

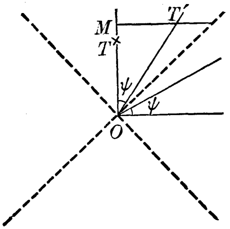

FIGURE 16.

Consider now (Figure 16) two intersecting (δ)-lines along which equal (unit) intervals OT and OT' are marked. If OT is taken as the time-axis, the point M, obtained by dropping from T' the perpendicular $T'M$ to OT, is simultaneous with T'. But the interval OM is greater than OT in the ratio $1 : \sqrt{1-u}$ where $u = \tanh \psi$ is the relative velocity determined by the two lines. Hence a unit time OT' as measured along OT' appears greater with reference to OT than the unit OT itself. This is another statement of Einstein's theorem that unit time, measured in a moving system, is longer than unit time measured in a stationary system.

All of these special thorems follow directly from the general transformation equations (7). We have

$$\begin{aligned} x_1' &= x_1 \cosh \psi - x_4 \sinh \psi, \\ x_4' &= -x_1 \sinh \psi + x_4 \cosh \psi. \end{aligned}$$

Now substituting

$$u = \tanh \psi, \qquad \sinh \psi = u / \sqrt{1-u^2}, \qquad \cosh \psi = 1 / \sqrt{1-u^2},$$

$$x_1' = \frac{1}{\sqrt{1-u^2}} (x_1 - u x_4),$$

$$x_4' = \frac{1}{\sqrt{1-u^2}} (x_4 - u x_1).$$

Or, replacing x_4 by t and x_1 by x, we have the fundamental transformation equations of Einstein for the change from stationary to moving coordinates.

20. Let us next consider instead of a (δ)-line any (δ)-curve. This will represent the space-time locus of a particle undergoing accelerated rectilinear motion. As the distinction between curved and straight lines is independent of any reference to axes, it follows that accelerated motion must remain accelerated motion regardless of the axes chosen. Moreover, the curvature (§ 17) of a curve is also independent of any choice of axes. Hence, although it is impossible, as we have seen, to define absolute velocity (that is, all velocity is relative to some assumed set of axes), we may define absolute acceleration if we are willing to define it as the curvature or as any function of the curvature alone. If, however, we wish to use the ordinary measure of acceleration, we must consider the projection of the curvature upon a chosen x-axis, namely,

$$c_x = \frac{1}{(1-v^2)^2}\frac{dv}{dt}, \quad \text{or} \quad \frac{dv}{dt} = (1-v^2)^2 c_x.$$

It is evident that curvature of constant magnitude does not mean uniform acceleration. Indeed if the numerical value of the curvature is constant the point in the xt-plane must move upon a pseudo-circle. Since the tangent to this curve approaches, but never reaches, the asymptotic fixed direction, it is clear that the velocity of the particle approaches as its limit the velocity of light. For such a motion, the relation between x and t is easily seen to be

$$(1-v^2)^{-\frac{3}{2}}\frac{dv}{dt} = \frac{1}{R}, \quad \text{or} \quad (x-c_2)^2 - (t-c_1)^2 = R^2,$$

where R is the radius of curvature, and c_1, c_2 are constants of integration depending on the choice of origin for x and t.

The interval of arc along any (δ)-curve is that which was called by Minkowski the Eigenzeit. This quantity is of course invariant in any change of axes. Thus

$$\int ds = \int \sqrt{dt^2 - dx^2} = \int \sqrt{dt'^2 - dx'^2}.$$

Mechanics of a Material Particle and of Radiant Energy.

21. Hitherto we have not assigned to our moving particles any distinguishing characteristics. Let us now consider what follows if we attribute to each particle a mass. It is true, as we shall later see, that the phenomena which must be discussed in connection with the dynamics of a material particle, even in the case where that particle moves only in a straight line, cannot be adequately represented in our two dimensional diagram. Nevertheless those results which can be discussed are so much more readily visualized in this simple case that we shall consider a few important theorems before entering upon the treatment of three and four dimensional manifolds.

The meaning of the mass of a particle, when that mass is determined by a person at rest relative to the particle, will be taken as understood. We shall call that value of the mass m_0. Let us consider a (δ)-curve which represents the locus in time and space of this material particle, and at any point of the locus a tangent of unit interval (or unit tangent) $\mathbf{w}$. By multiplying $\mathbf{w}$ by the scalar m_0, we make a new vector which we shall call the *extended momentum*. If now we choose any pair of axes x and t, the slope of the locus with respect to these axes, that is, the velocity of the particle, we have called v. The momentum vector may then be written, by (5),

$$m_0\mathbf{w} = \frac{m_0 v}{\sqrt{1 - v^2}}\mathbf{k}_1 + \frac{m_0}{\sqrt{1 - v^2}}\mathbf{k}_4. \tag{10}$$

If the t-axis were chosen parallel to the tangent $\mathbf{w}$, the coefficient of $\mathbf{k}_4$, that is, the component of the extended momentum $m_0\mathbf{w}$ along the time axis, would be simply m_0, the stationary mass. If, as we have assumed, the particle is regarded as moving with the velocity v, we shall take the component of $m_0\mathbf{w}$ along the t-axis as the mass m. In other words, the mass of a body appears to increase with its velocity in the familiar ratio

$$m = \frac{m_0}{\sqrt{1 - v^2}}. \tag{11}$$

The component along the x-axis is then mv, the momentum. We may therefore write the vector of extended momentum as

$$m_0\mathbf{w} = mv\mathbf{k}_1 + m\mathbf{k}_4. \tag{12}$$

22. From our equation for the curvature we may write

$$m_0\mathbf{c} = \frac{d\,m_0\mathbf{w}}{ds} = \frac{d\,mv}{ds}\mathbf{k}_1 + \frac{dm}{ds}\mathbf{k}_4 = \frac{1}{\sqrt{1 - v^2}}\left(\frac{d\,mv}{dt}\mathbf{k}_1 + \frac{dm}{dt}\mathbf{k}_4\right). \tag{13}$$

The vector $m_0\mathbf{c}$ we shall call the extended force. Since our ordinary definition of force is time-rate of change of momentum, it is evident that the x-component of the extended force multiplied by $\sqrt{1 - v^2}$ is ordinary force. That is,

$$f = \sqrt{1 - v^2}\, m_0 c_x = \frac{d\,mv}{dt}. \tag{14}$$

By comparison with equation (9), or by substituting for m from (11) and differentiating, we obtain the results[21]

$$f = \frac{dmv}{dt} = \frac{m_0 v}{(1 - v^2)^{\frac{3}{2}}} \frac{dv}{dt}, \tag{15}$$

$$\frac{dm}{dt} = \frac{m_0 v}{(1 - v^2)^{\frac{3}{2}}} \frac{dv}{dt} = fv = \frac{dE}{dt}, \tag{16}$$

where dE/dt represents the rate at which energy is acquired by the particle when acted upon by the force f. Since dE/dt and dm/dt are equal, we may, except possibly for a constant of integration, write $E = m$. This is a special statement which falls under the more general law, that the mass of a body, in the units which we employ, is equal to the energy of the body. We may therefore use the terms mass and energy interchangeably.

The type of motion which, from the viewpoint of the principle of relativity, corresponds most closely to motion under uniform acceleration in Newtonian mechanics, is motion under a constant force f. The equation of motion may readily be integrated.

$$f = \frac{dmv}{dt} = m_0 \frac{d}{dt} \frac{v}{\sqrt{(1 - v^2)}} = K,$$

$$\frac{v}{\sqrt{(1 - v^2)}} = \frac{K}{m_0}(t - t_0), \qquad \frac{dx}{dt} = \frac{Kt}{\sqrt{m_0^2 + K^2 (t - t_0)^2}},$$

and
$$\left(x - x_0 + \frac{m_0}{K}\right)^2 - (t - t_0)^2 = \frac{m_0^2}{K^2}.$$

The representative point in the xt-plane therefore describes a pseudo-circle of which the curvature is the constant force acting on the particle divided by m_0. The mass of the particle at any time is

$$m = \frac{m_0}{\sqrt{1 - v^2}} = \frac{K(t - t_0)}{v} = K\left(x - x_0 + \frac{m_0}{K}\right),$$

which shows that the increase in mass is equal to the product of the force by the distance traversed, as it should be from the principle of energy above stated.

23. Let us consider the problem of the impact of two particles A and B of which the vectors of extended momentum ($m_0\mathbf{w}$) are respectively **a** and **b** before collision, and **a**′ and **b**′ after collision. Several important laws are subsumed under a law which we may call the law of conservation of extended momentum, namely,

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[21] See later discussion (§36) of the so-called longitudinal mass.

$$\mathbf{a} + \mathbf{b} = \mathbf{a}' + \mathbf{b}'. \tag{17}$$

Assume any set of space-time axes, and write

$$\mathbf{a} = a_1\mathbf{k}_1 + a_4\mathbf{k}_4, \qquad \mathbf{b} = b_1\mathbf{k}_1 + b_4\mathbf{k}_4,$$

$$\mathbf{a}' = a_1'\mathbf{k}_1 + a_4'\mathbf{k}_4, \qquad \mathbf{b}' = b_1'\mathbf{k}_1 + b_4'\mathbf{k}_4.$$

Then the law states that

$$(a_1 + b_1)\,\mathbf{k}_1 + (a_4 + b_4)\,\mathbf{k}_4 = (a_1' + b_1')\,\mathbf{k}_1 + (a_4' + b_4')\,\mathbf{k}_4,$$

or

$$a_1 + b_1 = a_1' + b_1', \tag{18}$$

$$a_4 + b_4 = a_4' + b_4'. \tag{19}$$

Now (by § 21) a_4 and b_4 are the masses of the two particles before collision, a_4', b_4' the masses after collision, and equation (19) expresses the law of conservation of mass or energy. The components a_1, b_1, a_1', b_1', are the respective momenta (in the ordinary sense), and equation (18) is the law of conservation of momentum.

To assume that the impact is elastic is equivalent to assuming that the value of m_0 for each particle is unchanged by the collision; and since each value of m_0 is the magnitude of the corresponding vector of extended momentum, the assumption may be expressed in the equations

$$a = a', \qquad b = b'.$$

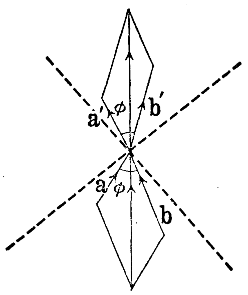

FIGURE 17.

The condition that the extended momentum is unchanged gives

$$(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) = (\mathbf{a}' + \mathbf{b}')\cdot(\mathbf{a}' + \mathbf{b}'),$$

or

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{b}'$$

by the above relations. Hence it follows (Figure 17) that

$$\cosh\phi = \cosh\phi', \qquad \text{or} \qquad \phi = \phi',$$

as is evident from the rules of projection previously deduced. It is thus seen that the relative velocity is the same before and after collision, and thereby a rule which has been found very useful in the discussion of simple problems in Newtonian mechanics proves equally applicable in the new mechanics.

If the impact, instead of being perfectly elastic, were such that the particles remained together after the collision, the two vectors **a** and **b** would merely be merged into a single vector $\mathbf{a} + \mathbf{b}$. The sum of the m_0's would not in this case remain constant, but would be increased by the heat (or mass) produced by the impact and obtained from the "kinetic energy" of the relative motion. This is all equivalent to the simple geometrical theorem that the (δ)-diagonal of a parallelogram whose sides are (δ)-lines is greater than the sum of the two sides.

24. The concepts of momentum and energy (mass) are ordinarily extended from the primitive mechanical phenomena to those involving so-called radiant energy. We shall see that the ascription of mass and momentum to light or other radiation is in consonance with the geometrical representation which we have adopted.

Let us consider a ray of light emitted in a single line for a definite interval of time. Such a ray alone can be considered in our two dimensional system. If the interval of time is very short, so that the front and the rear of the ray are very near together, we may regard the ray as a particle of light. The motion of such a light-particle can only be represented in our geometry by a singular vector, and to any observer its velocity is unity. Although the interval of any singular vector is zero as compared with the interval of any (γ)- or (δ)-vector, intervals along a given singular vector are, as we have pointed out, comparable with one another.

Supposing now that a given light-particle is represented by a definite singular vector, let us see whether such a vector can be regarded as an extended momentum. If so, its projection on any chosen space-axis must represent momentum, and its projection on the corresponding time-axis mass or energy. These two projections must, moreover, be of equal magnitude in this case, since the velocity of light is unity. It is immediately obvious that this latter condition is fulfilled, since the vector is singular (§ 11). If **a** is the vector, then in terms of two sets of axes

$$\mathbf{a} = m\mathbf{k}_1 + m\mathbf{k}_4 = m'\mathbf{k}_1' + m'\mathbf{k}_4'.$$

If then **a** represents extended momentum, m must represent the mass of the light to an observer stationary with respect to the first system of axes, and m' the mass as it appears to an observer stationary with respect to the other system.

If ϕ is the angle from $\mathbf{k}_1$ to $\mathbf{k}_1'$ or from $\mathbf{k}_4$ to $\mathbf{k}_4'$, we have from (7)

$$m' = m \cosh \phi - m \sinh \phi = m \frac{1 - v}{\sqrt{1 - v^2}}, \tag{20}$$

where $v = \tanh \phi$ is the relative velocity of the two sets of axes. But this is in fact the very relation between the energy of a given particle of light as measured by two different observers whose relative velocity is v. It is therefore, as far as the energy relations are concerned, proper to consider $\mathbf{a}$ as a vector of extended momentum.

The final proof of the desirability of considering the vector $\mathbf{a}$ as extended momentum comes when we consider the interaction of a light-particle with a particle of the ordinary sort. We shall see that the law of the constancy of extended momentum is true, and is only true, when we include the momentum of radiant energy as well as that of so-called material particles.

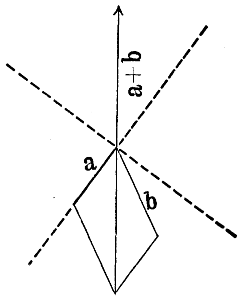

FIGURE 18.

Let the vector $\mathbf{a}$ (Figure 18) be the vector due to a light-particle, and $\mathbf{b}$ that due to a material particle which has the power of absorbing light. Then if our law of extended momentum applies to $\mathbf{a}$ and $\mathbf{b}$, there will be a single vector after impact equal to $\mathbf{a} + \mathbf{b}$ which will represent the extended momentum of the material particle after it has absorbed the light. Let us choose any set of axes. Then

$$\mathbf{a} = a_1 \mathbf{k}_1 + a_4 \mathbf{k}_4, \qquad \mathbf{b} = b_1 \mathbf{k}_1 + b_4 \mathbf{k}_4,$$

where $a_4 = a_4$ is the mass of the light-particle, and b_4 is the mass of the material particle before impact, while a_1 and $b_1 = b_4 v$ are the respective momenta. The momentum after impact is

$$a_1 + b_1 = a_4 + b_4 v.$$

Hence the change in momentum of the material particle is equal in our units to the energy of the light absorbed, which gives at once the well known formula of Maxwell and Boltzmann for the pressure of light.

While it is evident, therefore, that such a vector $\mathbf{a}$ satisfies fully all the conditions of an extended momentum, it must as a singular vector have properties quite distinct from those of a momentum vector which can be written in the form of $m_0 \mathbf{w}$. Since a singular vector has zero magnitude we can ascribe to the light no finite value of m_0 or $\mathbf{w}$. In this case, as in the case of inelastic impact between material particles, the total values of m_0 does not remain constant, but is larger after impact. In all cases we obtain the same results from the law of the constancy of extended momentum as those obtained by the application of the ordinary laws for the conservation of energy, mass, and momentum, whatever axes be arbitrarily chosen.

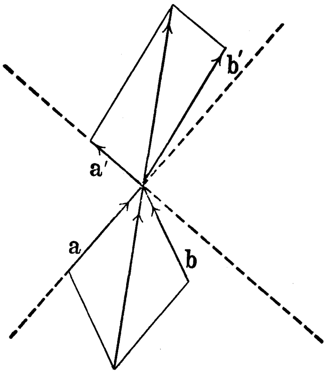

FIGURE 19.

Another simple illustration of these laws is furnished (Figure 19) in the case where the material particle does not absorb the light, but acts as a perfect reflector, which corresponds closely to elastic impact between particles. Here $\mathbf{a}'$ and $\mathbf{b}'$ are the vectors of the light-particle and the material particle after impact; and these vectors are readily shown to be determined either by the condition that the magnitude of $\mathbf{b}$ is equal to the magnitude of $\mathbf{b}'$, that is that the value of m_0 for the material particle undergoes no change, or from the condition that the angle between $\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ is the same as the angle between $\mathbf{b}'$ and $\mathbf{a}' + \mathbf{b}'$. This latter condition may in fact be regarded as necessary *à priori,* since it is the only construction which can be, in the nature of the case, uniquely determined.

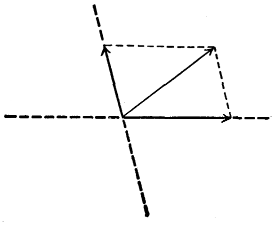

FIGURE 20.

Let us now consider light traveling back and forth in a single line between two mirrors whose positions are fixed relative to one another. If the mirrors are very close to one another, we may as before consider the whole system as concentrated at a point. This gives us a new kind of particle, an infinitesimal one-dimensional *Hohlraum.* Since however the energy contained within the particle is in part moving with the velocity of light in one direction and in part with the velocity of light in the other direction, we may draw two singular vectors (Figure 20) to represent the extended momenta in the two directions. Now these vectors added together give a (δ)-vector which will behave in every way like the extended momentum $m_0\mathbf{w}$ of a material particle, and m_0 represents the mass or energy of the *Hohlraum* as it appears to any observer at rest with respect to it. To such an observer the amount of energy traveling in one direction appears equal to that traveling in the opposite direction, and the resultant momentum is zero. To any observer moving with the velocity v relative to the particle, the momentum is the difference between the momenta which he observes in the two directions, and the mass of the particle is increased in the ratio $1/\sqrt{1 - v^2}$. These results are all evident geometrically, and follow analytically from (20).

THE NON-EUCLIDEAN GEOMETRY IN THREE DIMENSIONS.

Geometry, Outer and Inner Products.

25. We shall now consider a three-dimensional space in which the meaning of points, lines, planes, parallelism, and parallel-transformation or translation are precisely as in ordinary Euclidean geometry. In such a space, in addition to directed segments of lines or one-dimensional vectors, we have directed portions of planes or two-dimensional vectors. Any two portions of the same or parallel planes having the same area and the same sign will be considered identical two-dimensional vectors, briefly designated as 2-vectors. The ordinary one-dimensional vectors may be called 1-vectors for definiteness. It is evident that the outer product $\mathbf{a}\times\mathbf{b}$ of two 1-vectors in space is no longer a pseudo-scalar but a 2-vector lying in the plane determined by the two vectors and having a magnitude equal to the area of their parallelogram.

The addition of two 2-vectors may be accomplished geometrically in the following way. Take a definite segment of the line of intersection of the planes of the 2-vectors. In each plane construct on this segment as one side parallelograms equal respectively to the given 2-vectors. Complete the parallelepiped of which these two parallelograms are adjacent faces. The diagonal parallelogram of the parallelepiped, passing through the chosen segment, is the vector sum; the diagonal parallelogram parallel to the chosen segment is the vector difference.

Let us consider the outer product of a 1-vector and a 2-vector,[22] $\mathbf{a}\times\mathbf{A}$. Let $\mathbf{A}$ be represented as a parallelogram, and $\mathbf{a}$ as a vector through one vertex; the product $\mathbf{a}\times\mathbf{A}$ is the parallelepiped thus determined. This outer product $\mathbf{a}\times\mathbf{A}$, being three-dimensional in a three-dimensional space, is a pseudo-scalar; and different pseudo-scalars are distinguished only by magnitude and sign.

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[22] In general 2-vectors will be designated by Clarendon capitals (except in the case of the unit coordinate 2-vectors).

If in $\mathbf{a}\times\mathbf{A}$ we regard $\mathbf{A}$ as itself an outer product $\mathbf{b}\times\mathbf{c}$, the parallelepiped is written as $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. This same parallelepiped can be regarded, with the possible exception of sign, as $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. We shall in fact consider the sign as the same, and write

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\times\mathbf{c},$$

so that the associative law holds for the three factors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. As $\mathbf{b}\times\mathbf{c} = -\,\mathbf{c}\times\mathbf{b}$, we shall write $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -\,\mathbf{a}\times(\mathbf{c}\times\mathbf{b})$, in order that we may keep the law of association for the scalar factor. By successive steps we may write

$$\mathbf{a}\times\mathbf{b}\times\mathbf{c} = -\,\mathbf{b}\times\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}\times\mathbf{a};$$

and hence the outer product of a 1-vector and a 2-vector is not anti-commutative but commutative, namely,

$$\mathbf{a}\times\mathbf{A} = \mathbf{A}\times\mathbf{a}.$$

All of these statements are valid in any geometry of the group characterized by the parallel transformation.

26. In the three-dimensional non-Euclidean space, rotation about a fixed point is characterized by the existence of a fixed cone through the point, corresponding to the fixed lines in our plane geometry. An element of this cone always remains an element; points within the cone remain within, and points without remain outside. Besides the lines which are elements of this cone, or parallel to them, there are two classes, namely,

(δ)-lines through the vertex and lying within the cone, and all lines parallel to them,

(γ)-lines through the vertex and lying outside the cone, and all lines parallel to them.

In like manner planes may be separated into classes. Besides the planes of singular properties which are tangent to the cone along an element, or planes parallel to these, there are

(δ)-planes through the vertex cutting the cone in two elements, and all planes parallel thereto,

(γ)-planes through the vertex and not otherwise cutting the cone, and all parallel planes. The former set, the (δ)-planes, contain (δ)-

lines and also (γ)-lines; the latter set, the (γ)-planes, contain only (γ)-lines.

Any plane passed through a given (δ)-line cuts the cone in two elements and is therefore a (δ)-plane. The geometry of such a plane is the non-Euclidean plane geometry above described, and the elements of the cone are the fixed directions. The perpendicular in this plane to the given (δ)-line is a (γ)-line. The locus of the lines perpendicular to the given (δ)-line in all the planes through the line is a (γ)-plane. This (γ)-plane will be called perpendicular to the (δ)-line. Such a plane possesses no elements of the cone, that is, no lines which are fixed in rotation; hence the geometry of a (γ)-plane is ordinary Euclidean geometry. In the plane any line may be rotated into any other line, and the locus of the extremity of a given segment issuing from the center of rotation is a closed curve which is the circle in that plane. Moreover, the idea of angle, and of perpendicularity between lines in the (γ)-plane, being the same as in ordinary Euclidean geometry, need not be further defined.

A plane passed through a (γ)-line may cut the cone in two elements and be a (δ)-plane, or may fail to cut the cone and will then be a (γ)-plane.[23] The perpendiculars to a (γ)-line will therefore be in part (δ)-lines and in part (γ)-lines, and the plane perpendicular to a (γ)-line will therefore be a (δ)-plane. Thus a plane perpendicular to a (δ)-line is a (γ)-plane, and a plane perpendicular to a (γ)-line is a (δ)-plane.

In any three dimensional rotation one line, the axis of rotation, remains fixed, and points in any plane perpendicular to the axis remain in that plane. If the axis is a (δ)-line, the rotation is Euclidean; if a (γ)-line, non-Euclidean.

When all possible rotations, Euclidean and non-Euclidean, about axes through a given point are considered, the locus of the termini of a (γ)-vector of fixed interval, and a (δ)-vector of equal interval, issuing from the common center of the rotations, is a surface which from a completely Euclidean point of view appears to be the two conjugate hyperboloids of revolution asymptotic to the fixed cone, but which from our non-Euclidean viewpoint is really analogous to the sphere. The (δ)-lines cuts the two-parted hyperboloid; the (γ)-lines, the one-parted.

27. If we construct at a point three mutually perpendicular axes, two will be (γ)-lines, and one a (δ)-line. The unit vectors along these axes will be denoted respectively by $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_4$. The outer products $\mathbf{k}_1 \times \mathbf{k}_2$, $\mathbf{k}_1 \times \mathbf{k}_4$ $\mathbf{k}_2 \times \mathbf{k}_4$ will be denoted for brevity by $\mathbf{k}_{12}$, $\mathbf{k}_{14}$, $\mathbf{k}_{24}$.

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[23] Planes tangent to the cone will be discussed later.

In terms of these arbitrarily chosen axes a 1-vector may be represented as

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_4\mathbf{k}_4.$$

Similarly a 2-vector may be represented by the sum of its projections on the coordinate planes as

$$\mathbf{A} = A_{12}\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24}.$$

If we had chosen $\mathbf{k}_{21}$ in place of $\mathbf{k}_{12}$ as one of our unit coordinate 2-vectors, we should have written

$$\mathbf{A} = A_{21}\mathbf{k}_{21} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24}.$$

Since $A_{12}\,\mathbf{k}_{12} = A_{21}\,\mathbf{k}_{21}$ and $\mathbf{k}_{12} = -\,\mathbf{k}_{21}$, we have $A_{12} = -\,A_{21}$.

If we denote by $\mathbf{k}_{124}$ the outer product $\mathbf{k}_1 \times \mathbf{k}_2 \times \mathbf{k}_4$, then

$$\mathbf{k}_{124} = -\,\mathbf{k}_{142} = \mathbf{k}_{412} = -\,\mathbf{k}_{421} = \mathbf{k}_{241} = -\,\mathbf{k}_{214},$$

by the rules of outer products given above. In three-dimensional space these products are unit pseudo-scalars.

In terms of their components we may now expand the two types of outer product which occur in three-dimensional space. In this expansion we employ the distributive law and the law of association for scalar factors. Then

$$\mathbf{a} \times \mathbf{b} = (a_1b_2 - a_2b_1)\,\mathbf{k}_{12} + (a_1b_4 - a_4b_1)\,\mathbf{k}_{14} + (a_2b_4 - a_4b_2)\,\mathbf{k}_{24},$$
$$\mathbf{a} \times \mathbf{A} = (a_1A_{24} + a_2A_{41} + a_4A_{12})\mathbf{k}_{124}.$$

At this point we may discuss the general characteristics of inner and outer products of vectors of various geometric dimensionalities in an n-dimensional space. In such a space we have vectors of 0, 1, 2, . . ., n–1, n-dimensions, designated as 0-vectors (or scalars), 1-vectors, 2-vectors, . . ., (n–1)-vectors, and n-vectors (or pseudo-scalars). The outer product of a p-vector and a q-vector is a $(p + q)$-vector; the product vanishes if by translation the p-vector and q-vector can be made to lie in space of less than $p + q$ dimensions. The inner product of a p-vector and a q-vector, where $p \geqq q$, will always be defined as a $(p-q)$-vector. Thus whereas the inner product of a 1-vector by a 1-vector is a scalar, the inner product of a 1-vector and a 2-vector is a 1-vector.

Both the inner and outer products will obey the distributive law, and the associative law as far as regards multiplication by a scalar factor. Furthermore the outer product will always obey the associative law, and the inner product the commutative law.

28. The inner product of any 1-vector into itself may, by an immediate generalization of the definition in plane geometry (§ 14), be defined as equal to the square of its interval, taken positively for (γ)-vectors, negatively for (δ)-vectors. The inner product of two 1-vectors is equal to the inner product of either one and the projection of the other upon it. The rules for the unit coordinate vectors are therefore

$$\mathbf{k}_1 \cdot \mathbf{k}_1 = \mathbf{k}_2 \cdot \mathbf{k}_2 = 1, \qquad \mathbf{k}_4 \cdot \mathbf{k}_4 = -1, \qquad \mathbf{k}_1 \cdot \mathbf{k}_2 = \mathbf{k}_1 \cdot \mathbf{k}_4 = \mathbf{k}_2 \cdot \mathbf{k}_4 = 0.$$

The product of two vectors

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_4\mathbf{k}_4, \qquad \mathbf{b} = b_1\mathbf{k}_1 + b_2\mathbf{k}_2 + b_4\mathbf{k}_4,$$

is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 - a_4 b_4.$$

The inner product $\mathbf{a} \cdot \mathbf{A}$ of a 1-vector and a 2-vector will be a 1-vector in the plane **A** and perpendicular to **a** (that is, perpendicular to the projection of **a** on **A**); its magnitude will be equal to the product of the magnitude of **A** and the magnitude of the projection of **a** on **A**; its sign is best determined analytically. If **a** and **b** are perpendicular 1-vectors we may make the convention

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \mathbf{a}(\mathbf{b} \cdot \mathbf{b}), \qquad \text{or} \qquad (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = -\mathbf{b}(\mathbf{a} \cdot \mathbf{a}). \tag{21}$$

Thence follow the rules for the unit vectors,

$$\begin{array}{lll}
\mathbf{k}_1 \cdot \mathbf{k}_{12} = -\mathbf{k}_2, & \mathbf{k}_1 \cdot \mathbf{k}_{14} = -\mathbf{k}_4, & \mathbf{k}_1 \cdot \mathbf{k}_{24} = 0, \\
\mathbf{k}_2 \cdot \mathbf{k}_{12} = \mathbf{k}_1, & \mathbf{k}_2 \cdot \mathbf{k}_{14} = 0, & \mathbf{k}_2 \cdot \mathbf{k}_{24} = -\mathbf{k}_4, \\
\mathbf{k}_4 \cdot \mathbf{k}_{12} = 0, & \mathbf{k}_4 \cdot \mathbf{k}_{14} = -\mathbf{k}_1, & \mathbf{k}_4 \cdot \mathbf{k}_{24} = -\mathbf{k}_2.
\end{array}$$

Hence [24]

$$\mathbf{a} \cdot \mathbf{A} = (a_2 A_{12} - a_4 A_{14})\,\mathbf{k}_1 + (-a_1 A_{12} - a_4 A_{24})\,\mathbf{k}_2 + (-a_1 A_{14} - a_2 A_{24})\,\mathbf{k}_4.$$

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[24] We may show that these rules do give an inner product which in all cases agrees with the geometric definition above stated.

The condition that $\mathbf{a} \cdot \mathbf{A}$ lies in the plane **A** is that the outer product of it and **A** shall vanish, that is, $(\mathbf{a} \cdot \mathbf{A}) \times \mathbf{A} = 0$; the condition that it is perpendicular to **a** is that the inner product of it and **a** shall vanish, that is, $(\mathbf{a} \cdot \mathbf{A}) \cdot \mathbf{a} = 0$. These two products are

$$\begin{aligned}(\mathbf{a} \cdot \mathbf{A}) \times \mathbf{A} = [(a_2 A_{12} - a_4 A_{14}) A_{24} + (a_1 A_{12} + a_4 A_{24}) A_{14} \\ - (a_1 A_{14} + a_2 A_{24}) A_{12}]\,\mathbf{k}_{124} = 0,\end{aligned}$$

$$(\mathbf{a} \cdot \mathbf{A}) \cdot \mathbf{a} = a_1 (a_2 A_{12} - a_4 A_{14}) - a_2 (a_1 A_{12} + a_4 A_{24}) + a_4 (a_1 A_{14} + a_2 A_{24}) = 0,$$

as required. It is also necessary to show that the component of **a** perpendicular to **A** contributes nothing to the product $\mathbf{a} \cdot \mathbf{A}$, so that the component in

The inner product of two 2-vectors is a scalar which is equal to the inner product of either vector by the projection of the other upon it. The inner product of two perpendicular 2-vectors is zero. The inner product of a 2-vector by itself is numerically equal to the square of its magnitude, and is positive in sign if the vector is of class (γ), negative if of class (δ). Hence we have as rules of inner multiplication for 2-vectors

$$\mathbf{k}_{12}\cdot\mathbf{k}_{12} = 1, \qquad \mathbf{k}_{14}\cdot\mathbf{k}_{14} = \mathbf{k}_{24}\cdot\mathbf{k}_{24} = -1,$$

$$\mathbf{k}_{12}\cdot\mathbf{k}_{14} = \mathbf{k}_{12}\cdot\mathbf{k}_{24} = \mathbf{k}_{14}\cdot\mathbf{k}_{24} = 0,$$

$$\mathbf{A}\cdot\mathbf{A} = A_{12}^2 - A_{14}^2 - A_{24}^2, \qquad \mathbf{A}\cdot\mathbf{B} = A_{12}B_{12} - A_{14}B_{14} - A_{24}B_{24}.$$

29. Every 1-vector $\mathbf{a}$, or 2-vector $\mathbf{A}$ in a three-dimensional space uniquely determines, except for sign, another vector (respectively a 2-vector or 1-vector) perpendicular to it and of equal magnitude. This vector will be called the complement of the given vector, and designated as $\mathbf{a}^*$ or $\mathbf{A}^*$ respectively. To specify the sign, the complement may be defined as the inner product of the vector $\mathbf{a}$ or $\mathbf{A}$ and the unit 3-vector or pseudo-scalar $\mathbf{k}_{124}$, where the laws of this inner product are

$$\mathbf{k}_1\cdot\mathbf{k}_{124} = \mathbf{k}_{24}, \qquad \mathbf{k}_2\cdot\mathbf{k}_{124} = -\mathbf{k}_{14}, \qquad \mathbf{k}_4\cdot\mathbf{k}_{124} = -\mathbf{k}_{12},$$

$$\mathbf{k}_{12}\cdot\mathbf{k}_{124} = \mathbf{k}_4, \qquad \mathbf{k}_{14}\cdot\mathbf{k}_{124} = \mathbf{k}_2, \qquad \mathbf{k}_{24}\cdot\mathbf{k}_{124} = -\mathbf{k}_1.$$

Thus

$$\mathbf{a}^* = (a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_4\mathbf{k}_4)\cdot\mathbf{k}_{124} = -a_4\mathbf{k}_{12} - a_2\mathbf{k}_{14} + a_1\mathbf{k}_{24},$$

$$\mathbf{A}^* = (A_{12}\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24})\cdot\mathbf{k}_{124} = -A_{24}\mathbf{k}_1 + A_{14}\mathbf{k}_2 + A_{12}\mathbf{k}_4.$$

These complements satisfy the condition of perpendicularity previously derived (footnote 24), and the inner products

$$\mathbf{a}^*\cdot\mathbf{a}^* = a_4^2 - a_2^2 - a_1^2, \qquad \mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 - a_4^2,$$

$$\mathbf{A}^*\cdot\mathbf{A}^* = A_{24}^2 + A_{14}^2 - A_{12}^2, \qquad \mathbf{A}\cdot\mathbf{A} = A_{12}^2 - A_{14}^2 - A_{24}^2$$

---

the plane is alone of importance. We shall do this by deriving the expression for a vector perpendicular to the plane $\mathbf{A}$. Let

$$\mathbf{c} = c_1\,\mathbf{k}_1 + c_2\,\mathbf{k}_2 + c_4\,\mathbf{k}_4, \quad \mathbf{n} = n_1\,\mathbf{k}_1 + n_2\mathbf{k}_2 + n_4\,\mathbf{k}_4$$

be respectively any vector in the plane $\mathbf{A}$ and a vector perpendicular to the plane. Then the products

$$\mathbf{c}\times\mathbf{A} = (c_1A_{24} - c_2A_{14} + c_4A_{12})\mathbf{k}_{124} = 0, \qquad \mathbf{c}\cdot\mathbf{n} = c_1\,n_1 + c_2\,n_2 - c_4\,n_4 = 0$$

vanish. Hence it follows that the condition of perpendicularity for the vectors $\mathbf{n}$ and $\mathbf{A}$ is

$$n_1 : n_2 : n_4 = A_{24} : -A_{14} : -A_{12},$$

and that $\mathbf{n}$ must be some multiple of $A_{24}\mathbf{k}_1 - A_{14}\mathbf{k}_2 - A_{12}\mathbf{k}_4$. By the rules, the inner product of this vector and $\mathbf{A}$ vanishes.

show that the magnitudes are equal. The reversal of sign is to be expected from the fact that the complement of a vector (whether 1- or 2-) of class (γ) is a (δ)-vector (whether 2- or 1-), and vice versa.

The use of the term complement in connection with scalars and pseudo-scalars is sometimes convenient. Since, by the rule of inner multiplication, we have $\mathbf{k}_{124}\cdot\mathbf{k}_{124} = -1$, the complement of any pseudo-scalar is a scalar of the same magnitude and of opposite sign. We may define the complement of a scalar a as the product of the scalar and the unit pseudo-scalar; thus $a^* = a\mathbf{k}_{124}$.

All the special rules for the inner products of unit vectors (and pseudo-scalars) are comprised in the following general rule, which will also be applied in space of four dimensions: If either of two unit vectors has a subscript which the other lacks, the inner product is zero; in all other cases the inner product may be found by so transposing the subscripts that all the common subscripts occur in each factor at the end, and in the same order, by then canceling the common subscripts, and by taking as the product the unit vector which has the remaining subscripts (in the order in which they stand), provided that if the subscript 4 has been canceled, the sign is changed.[25] Thus

$$\mathbf{k}_{124}\cdot\mathbf{k}_{34} = 0, \quad \mathbf{k}_{124}\cdot\mathbf{k}_{12} = \mathbf{k}_{412}\cdot\mathbf{k}_{12} = \mathbf{k}_4, \quad \mathbf{k}_{12}\cdot\mathbf{k}_1 = -\mathbf{k}_{21}\cdot\mathbf{k}_1 = -\mathbf{k}_2,$$
$$\mathbf{k}_{124}\cdot\mathbf{k}_4 = -\mathbf{k}_{12}, \quad \mathbf{k}_{134}\cdot\mathbf{k}_{14} = -\mathbf{k}_{314}\cdot\mathbf{k}_{14} = \mathbf{k}_3.$$

30. Hitherto we have given little attention to the singular vectors of our geometry, namely, the lines which are elements of a singular cone and the planes which are tangent to a singular cone. We have seen (§ 14) that the inner product of a singular 1-vector by itself is zero, and have expressed that fact by stating that a singular line is perpendicular to itself. Analytically expressed, the condition that a 1-vector $\mathbf{a}$ shall be singular is that

$$\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 - a_4^2 = 0.$$

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[25] Instead of regarding the common subscripts as canceled, it is possible to regard their corresponding unit 1-vectors as multiplied by inner multiplication,— and in this case the change of sign takes care of itself. Thus

$$\mathbf{k}_{pqr}\cdot\mathbf{k}_{qr} = \mathbf{k}_p(\mathbf{k}_q\cdot\mathbf{k}_q)(\mathbf{k}_r\cdot\mathbf{k}_r).$$

Indeed if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are mutually perpendicular 1-vectors, then all the rules given above may be expressed in the equations

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}), \qquad (\mathbf{a}\times\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{c}),$$
$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \mathbf{a}(\mathbf{b}\cdot\mathbf{b}), \qquad (\mathbf{a}\times\mathbf{b}\times\mathbf{c})\cdot\mathbf{c} = \mathbf{a}\times\mathbf{b}(\mathbf{c}\cdot\mathbf{c}),$$
$$(\mathbf{a}\times\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}(\mathbf{b}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{c}).$$

Thus any singular vector may be written in the form

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 \pm \sqrt{a_1^2 + a_2^2}\,\mathbf{k}_4.$$

The complement of a singular vector is

$$\mathbf{A} = \mathbf{a}^* = \mathbf{a}\cdot\mathbf{k}_{124} = a_1\mathbf{k}_{24} - a_2\mathbf{k}_{14} \mp \sqrt{a_1^2 + a_2^2}\,\mathbf{k}_{12}.$$

This 2-vector **A** must be itself a singular plane vector; for we have seen that the complement of any (δ)-plane is a (γ)-line and of any (γ)-plane a (δ)-line, and vice versa. The inner product of **A** by itself is obviously zero,[26] for,

$$\mathbf{A}\cdot\mathbf{A} = -a_1^2 - a_2^2 + (a_1^2 + a_2^2) = 0.$$

Conversely if we consider any 2-vector

$$\mathbf{A} = A_{12}\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24},$$

such that

$$\mathbf{A}\cdot\mathbf{A} = A_{12}^2 - A_{14}^2 - A_{24}^2 = 0,$$

its complement is a singular line, and it is itself a singular 2-vector. The standard form may be taken as

$$\mathbf{A} = \pm\sqrt{A_{14}^2 + A_{24}^2}\,\mathbf{k}_{12} + A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24}.$$

The outer product of a singular vector by its complement, whether a 1-vector or a 2-vector, vanishes, as may be seen by multiplying out. Thus the singular vector and its complement lie in the same plane, that is, an element of the cone and the tangent plane through that element are mutually complementary.

When we have to consider the inner product of any singular vector with any other vector, singular or not, the geometrical method dependent on projection often fails to be applicable; for it is impossible to project a vector upon a singular vector. We may in such cases employ the analytical method, which is universally applicable, or replace the inner product with an outer product by a method introduced in a following section (§ 32).

We have seen that an element of the cone is complementary to the tangent plane to the cone through that element, that is, the element is perpendicular to the plane. Hence the element is perpendicular to every line in the plane (including itself).

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[26] A singular vector, or vector of zero magnitude, has, like any other vector, a real geometrical existence and is not to be confused with a zero vector, that is, a non-existent vector.

31. We have seen that rotation in a (γ)-plane about the perpendicular (δ)-line is Euclidean, whereas rotation in a (δ)-plane about the normal (γ)-line is non-Euclidean. In this latter case not only do the (δ)-planes normal to the axis remain fixed during the rotation, but the *two singular planes* through the axis and tangent to the cone also are fixed; for the axis remains fixed and the lines in which the planes are tangent to the cone are respectively the two fixed lines in the (δ)-plane. As every point in the axis of rotation is fixed, the whole set of lines parallel to the elements of tangency is fixed. The effect in the two singular planes of a rotation is therefore to leave one line, the axis, fixed point for point, to leave a set of lines fixed, and to move the points on these lines either toward the axis or away from it by an amount which is proportional to the interval from the point to the axis.

Since a rotation in a (δ)-plane multiplies all intervals along one of the fixed directions in a certain ratio, and divides all intervals along the other fixed direction in the same ratio, the effect upon areas in the two singular planes is to multiply all areas in one of the planes in that same ratio, and to divide areas in the other in that ratio. This however is not inconsistent with our condition that areas should remain invariant; for it is evident that, when compared with areas in other planes, areas in singular planes are all of zero magnitude. This is also shown by the fact that the inner product of any singular vector by itself vanishes. That areas in a singular plane have a zero magnitude does not prevent our comparing two areas in the same singular plane or in parallel singular planes, just as the fact that intervals along singular lines had zero magnitude did not prevent our comparing intervals along any such line.

A limiting case of rotation occurs when the axis of rotation is itself an element of the cone, that is, a singular line. Here the infinity of fixed planes perpendicular to the axis, and the two singular planes through it, have all coalesced into the one singular plane through this line and tangent to the cone. In this plane the rotation consists in a sort of shear. Every point moves along a straight line parallel to the axis. In this case areas are rotated into areas which are from every point of view equal. For if a parallelogram whose base is on the axis, which is fixed point for point, is subjected to this rotation, its base remains fixed and the parallelogram remains enclosed between the same two parallel lines (Theorem IX).

The geometry in this plane, depending upon translation and upon such a rotation as has just been described, is interesting as affording a third geometry intermediate between the Euclidean and the non-Euclidean which we have discussed. In Euclidean plane geometry there is no line fixed in rotation, in our non-Euclidean plane geometry there are two fixed directions, in this new case there is just one. If we were to investigate this geometry, we should find one set of (parallel) singular lines and one set of non-singular lines. Every non-singular line may be rotated into any other. Angles about any point range from $-\infty$ to $+\infty$ on each side of the singular line through that point. The interval along any line intercepted between two singular lines is equal to the interval along any other line thus intercepted. Every non-singular line is perpendicular to the singular lines, as the singular line is complementary to the singular plane through it.

Some Algebraic Rules.

32. We shall develop here a number of important relations between outer products, inner products, and complements which will be of frequent use later. Many of these relations hold in any number of dimensions. We shall consider primarily a non-Euclidean space in which one of a set of mutually perpendicular lines is a (δ)-line, the rest being (γ)-lines. But except for occasional differences of sign, the results are valid in a Euclidean space.

In a space of n dimensions, the complement of a vector of dimensionality p is itself of dimensionality $n-p$. If a is a scalar and α is a vector of any dimensionality, then from the associative law for scalar factors, we have

$$(a\alpha)^* = (a\alpha)\cdot\mathbf{k}_{12\ldots n} = (a\mathbf{k}_{12\ldots n})\cdot\alpha = a(\alpha\cdot\mathbf{k}_{12\ldots n}) = a^*\cdot\alpha = a\alpha^*. \quad (22)$$

Let α, β, . . . be vectors of the respective dimensionalities p, q. . . . Then

$$\beta\times\alpha = (-1)^{pq}\,\alpha\times\beta. \quad (23)$$

Owing to the availability of the distributive laws it is sufficient to prove such relations as this for the simpler case where the constituent vectors α, β are unit vectors $\mathbf{k}_{g\ldots}$, $\mathbf{k}_{h\ldots}$ of dimensionality p, q. In the permutation of α and β, there are involved pq simple transpositions of subscripts; for each subscript in $\mathbf{k}_{h\ldots}$ has to be carried past all the subscripts of $\mathbf{k}_{g\ldots}$ Hence there are pq changes of sign. Hence the outer product is commutative if either of the factors is even, but is anti-commutative if both factors are odd in dimensionality.

We may next show that

$$(\alpha\times\beta)^* = \alpha\cdot\beta^*. \tag{24}$$

Suppose again that α, β are unit vectors $\mathbf{k}_g\ldots$, $\mathbf{k}_h\ldots$. We have to show

$$(\mathbf{k}_g\ldots\times\mathbf{k}_h\ldots)\cdot\mathbf{k}_1\ldots = \mathbf{k}_g\ldots\cdot(\mathbf{k}_h\ldots\cdot\mathbf{k}_1\ldots),$$

where $\mathbf{k}_1\ldots$ denotes the unit pseudo-scalar. Without changing this equation, it is possible on both sides to arrange at the end, the subscripts of the pseudo-scalar $\mathbf{k}_1\ldots$ in the same order as in the factors $\mathbf{k}_g\ldots$, $\mathbf{k}_h\ldots$ Thus we have to show that

$$(\mathbf{k}_g\ldots\times\mathbf{k}_h\ldots)\cdot\mathbf{k}_{j\ldots g\ldots h\ldots} = \mathbf{k}_g\ldots\cdot(\mathbf{k}_h\ldots\cdot\mathbf{k}_{j\ldots g\ldots h\ldots}).$$

But now the products on the right are found by canceling successively the common subscripts $h\ldots$ and $g\ldots$; whereas the product on the left is found by canceling simultaneously the subscripts of $\mathbf{k}_{g\ldots h\ldots}$. The identity is therefore proved.

As a corollary of the two preceding results we may write the formula

$$(\alpha\times\beta)^* = (-1)^{pq}(\beta\times\alpha)^* = \alpha\cdot\beta^* = (-1)^{pq}\beta\cdot\alpha^*. \tag{25}$$

All these rules are true for any space, Euclidean or non-Euclidean.

The complement of the complement of a vector α is the vector itself, except for sign. If α is of dimensionality p in a space of n dimensions, the exact relation is

$$(\alpha^*)^* = -(-1)^{p(n-p)}\alpha. \tag{26}$$

The complement of the complement of a vector will therefore be the negative of the vector except when $p\,(n-p)$ is odd, that is, when the dimensionalities of the vector and of the space are respectively odd and even.[27] For the proof, the consideration may be restricted to the case where α is a unit vector $\mathbf{k}_g\ldots$. Then

$$\begin{aligned}(\alpha^*)^* = (\mathbf{k}_g\ldots\cdot\mathbf{k}_i\ldots)\cdot\mathbf{k}_i\ldots &= (\mathbf{k}_g\ldots\cdot\mathbf{k}_{j\ldots g\ldots})\cdot\mathbf{k}_{j\ldots g\ldots}\\ &= (-1)^{p(n-p)}(\mathbf{k}_g\ldots\cdot\mathbf{k}_{j\ldots g\ldots})\cdot\mathbf{k}_{g\ldots j\ldots}\end{aligned}$$

Here again the subscripts in the pseudo-scalar $\mathbf{k}_i\ldots$ have been rearranged so as to bring $g\ldots$ to the end. Then as $g\ldots$ denotes p subscripts and $j\ldots$ denotes $n-p$, the permutation involves $p\,(n-p)$

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[27] In Euclidean space $(\alpha^*)^* = (-1)^{p(n-p)}\alpha$. Some writers who have identified vectors with their complements have perhaps overlooked this relation which would, upon their assumption, make a vector sometimes identical with its own negative.

changes of sign. In the final form thus found the subscripts $g \dots$ and $j \dots$ have successively to be canceled. But one of these is necessarily the subscript 4 (corresponding to the (δ)-vector), which requires a change of sign. Hence

$$(\mathbf{k}_g\dots\cdot\mathbf{k}_i\dots)\cdot\mathbf{k}_i\dots = -(-1)^{p(n-p)}\mathbf{k}_g\dots,$$

and the desired result is proved.

Consider the product $\alpha^*\cdot\beta^*$. We have by (24) either

$$\alpha^*\cdot\beta^* = (\alpha^*\times\beta)^* \qquad \text{or} \qquad \beta^*\cdot\alpha^* = (\beta^*\times\alpha)^*. \tag{27}$$

Now, although $\alpha^*\cdot\beta^*$ and $\beta^*\cdot\alpha^*$ are equal, the two expansions obtained are usually different. In fact, as the total dimensionality of an outer product cannot exceed n, the first formula holds only when $p \geqq q$, and the second only when $q \geqq p$. Let us assume $q \geqq p$. Then

$$\begin{aligned}\alpha^*\cdot\beta^* = \beta^*\cdot\alpha^* = (\beta^*\times\alpha)^* &= (-1)^{p(n-q)}(\alpha\times\beta^*)^* \\ &= (-1)^{p(n-q)}\alpha\cdot\beta^{**} = -(-1)^{(q-p)(n-q)}\alpha\cdot\beta.\end{aligned} \tag{28}$$

As a corollary

$$\alpha^*\cdot\alpha^* = -\alpha\cdot\alpha. \tag{29}$$

The complement of an inner product may likewise be proved to be

$$(\alpha\cdot\beta)^* = (-1)^{p(n-p)}\alpha\times\beta^*, \tag{30}$$

where it is assumed that the product $\alpha\cdot\beta$ has been so arranged that the second factor is of dimensionality q greater than the dimensionality p of the first. We have furthermore

$$\alpha^*\times\alpha = (\alpha\cdot\alpha)^*; \tag{31}$$

and also if β is a pseudo-scalar

$$(\alpha\cdot\beta)^* = (-1)^{p(n-p)}\beta^*\alpha = \beta\cdot\alpha^*. \tag{32}$$

It is important to observe that by means of these rules it is possible to replace any outer product by an inner product, and vice versa.

33. We are now able to obtain rules for the expansion of the various products in which three vectors occur. The simplest type, and one which needs no further comment, is

$$(\alpha\times\beta)\times\gamma = \alpha\times(\beta\times\gamma), \tag{33}$$

which follows from the associative law.

Consider next the product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ of three 1-vectors. Here

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}. \tag{34}$$

Perhaps the simplest proof is obtained from the relation [28]

$$\mathbf{b} = \frac{(\mathbf{b}\cdot\mathbf{c})\,\mathbf{c}}{\mathbf{c}\cdot\mathbf{c}} + \frac{\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})}{\mathbf{c}\cdot\mathbf{c}},$$

which states that a vector is equal to the sum of its components. By clearing and transposing, and by permuting the letters, we have

$$\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\,\mathbf{c},$$

$$\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\,\mathbf{c}.$$

If now $\mathbf{d}$ is any vector perpendicular to $\mathbf{b}$ and $\mathbf{c}$, we have identically

$$\mathbf{d}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{d}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{d}\cdot\mathbf{b})\,\mathbf{c} = 0.$$

If these equations be multiplied by x, y, z and added, we have

$$(x\mathbf{c} + y\mathbf{b} + z\mathbf{d})\cdot(\mathbf{b}\times\mathbf{c}) = [(x\mathbf{c} + y\mathbf{b} + z\mathbf{d})\cdot\mathbf{c}]\mathbf{b} - [(x\mathbf{c} + y\mathbf{b} + z\mathbf{d})\cdot\mathbf{b}]\mathbf{c},$$

and any vector $\mathbf{a}$ may be represented in the form $x\mathbf{c} + y\mathbf{b} + z\mathbf{d}$.

From the rules (33), (34) combined with the rules (22)–(32) we may obtain a number of other reduction formulas by simply taking complements of both sides of the equation.

Thus

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{C} = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{C}) = -\mathbf{b}\cdot(\mathbf{a}\cdot\mathbf{C}). \tag{35}$$

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[28] With the aid of inner and outer products we may write down expressions for the components of a 1-vector $\mathbf{a}$ along and perpendicular to another 1-vector $\mathbf{b}$ or a 2-vector $\mathbf{A}$. The components of $\mathbf{a}$ along $\mathbf{b}$ and perpendicular to $\mathbf{b}$ are

$$\frac{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}} \quad\text{and}\quad \frac{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}.$$

The components of $\mathbf{a}$ along $\mathbf{A}$ and perpendicular to $\mathbf{A}$ are

$$-\frac{(\mathbf{a}\cdot\mathbf{A})\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}} \quad\text{and}\quad \frac{(\mathbf{a}\times\mathbf{A})\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}.$$

The component of the plane $\mathbf{A}$ on the plane $\mathbf{B}$ is

$$\frac{(\mathbf{A}\cdot\mathbf{B})\,\mathbf{B}}{\mathbf{B}\cdot\mathbf{B}},$$

and a vector in the line of intersection of the two planes is

$$\mathbf{A}^*\cdot\mathbf{B} \quad\text{or}\quad \mathbf{A}\cdot\mathbf{B}^*.$$

For by (33) and (24),

$$[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]^* = [\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]^*,$$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}^* = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})^* = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}^*).$$

But since $\mathbf{c}$ is any 1-vector, its complement $\mathbf{C}$ is any 2-vector.

Again,

$$\mathbf{a}\times(\mathbf{b}\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{C}. \tag{36}$$

For by (34), (22), and (30),

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^* = [(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}]^* - [(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]^*,$$

$$(-1)^{1(3-1)}\,\mathbf{a}\times(\mathbf{b}\times\mathbf{c})^* = (-1)^{1(3-1)}\,(\mathbf{a}\times\mathbf{c}^*)\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}^*,$$

$$\mathbf{a}\times(\mathbf{b}\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{C}.$$

Again,

$$(\mathbf{b}\cdot\mathbf{C})\times\mathbf{A} = -\,\mathbf{C}\times(\mathbf{b}\cdot\mathbf{A}). \tag{37}$$

For from (35), (30), and (24),

$$[\mathbf{C}\cdot(\mathbf{a}\times\mathbf{b})]^* = [(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{a}]^*,$$

$$(-1)^{2(3-2)}\,\mathbf{C}\times(\mathbf{a}\times\mathbf{b})^* = (-1)^{1(3-1)}\,(\mathbf{b}\cdot\mathbf{C})\times\mathbf{a}^*,$$

$$-\,\mathbf{C}\times(\mathbf{b}\times\mathbf{a})^* = -\,\mathbf{C}\times(\mathbf{b}\cdot\mathbf{A}) = (\mathbf{b}\cdot\mathbf{C})\times\mathbf{A}.$$

Again

$$(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A} = -\,\mathbf{b}\,(\mathbf{C}\cdot\mathbf{A}) + \mathbf{C}\cdot(\mathbf{b}\times\mathbf{A}). \tag{38}$$

For from (36), (24), (32), (22), and (30),

$$[(\mathbf{b}\cdot\mathbf{C})\times\mathbf{a}]^* = -\,[\mathbf{b}\cdot(\mathbf{C}\times\mathbf{a})]^* + [\mathbf{C}\,(\mathbf{b}\cdot\mathbf{a})]^*,$$

$$(\mathbf{b}\cdot\mathbf{C})\cdot\mathbf{A} = -\,\mathbf{b}\,(\mathbf{C}\times\mathbf{a})^* + \mathbf{C}\cdot(\mathbf{b}\cdot\mathbf{a})^*,$$

$$= -\,\mathbf{b}(\mathbf{C}\cdot\mathbf{A}) + (-1)^{1(3-1)}\,\mathbf{C}\cdot(\mathbf{b}\times\mathbf{A}).$$

These rules (33) to (38) involve every possible combination of three vectors in three dimensional space. Since the formulas which we have used in deriving them, have the same form in Euclidean space, the rules will be true in Euclidean space. The particular use of the complement has implied a three dimensional space, and a similar use of the complenent in a four dimensional space would obtain analogous but different formulas; it should be observed, however, that the rules here obtained (with the exception of (37)) must hold in space of four dimensions, even when the three vectors in question do not lie wholly in a three dimensional space. For consider (36) as a typical case. Let $\mathbf{b}$ be a 1-vector which does not lie in the space of $\mathbf{a}$ and $\mathbf{C}$; we may write $\mathbf{b} = \mathbf{b}' + \mathbf{b}''$, where $\mathbf{b}'$ is in the space of $\mathbf{a}$ and $\mathbf{C}$ and $\mathbf{b}''$ is perpendicular to $\mathbf{a}$ and $\mathbf{C}$. Then by (36)

$$\mathbf{a}\times(\mathbf{b}'\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b}' - (\mathbf{a}\cdot\mathbf{b}')\,\mathbf{C},$$

and

$$\mathbf{a}\times(\mathbf{b}''\cdot\mathbf{C}) = (\mathbf{a}\times\mathbf{C})\cdot\mathbf{b}'' - (\mathbf{a}\cdot\mathbf{b}'')\mathbf{C}$$

holds identically, since each of its terms vanishes. Hence by addition (36) is seen also to hold in general.

Some products involving more than three 1-vectors are of frequent occurrence. By (35) and (34) we may write immediately

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}. \quad (39)$$

In a similar manner we may prove

$$(\mathbf{a}\times\mathbf{b}\times\mathbf{c})\cdot(\mathbf{d}\times\mathbf{e}\times\mathbf{f}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{e} & \mathbf{a}\cdot\mathbf{f} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{f} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{f} \end{vmatrix},$$

and

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}\times\mathbf{e}) = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{d}\times\mathbf{e})\,\mathbf{c} + (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{e}\times\mathbf{c})\,\mathbf{d} + (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})\,\mathbf{e}.$$

These formulas are all valid in space of any dimensions.

The Differentiating Operator ∇.

34. In discussing the differential calculus of scalar and vector functions of position in space, the vector differentiating operator ∇ is fundamental. The definition of this operator may be most simply obtained as follows. Consider a scalar function F of position in space. Let $d\mathbf{r}$ denote any infinitesimal vector change of position, and let dF denote the corresponding differential change in F. Then let ∇ be defined by the equation

$$dF = d\mathbf{r}\cdot\nabla F.$$

Now ∇F is a vector. If $d\mathbf{r}$ is a vector in the tangent plane to the surface $F =$ const., dF is 0, and as $d\mathbf{r}\cdot\nabla F$ then vanishes, the vector $d\mathbf{r}$ and ∇F are perpendicular. Hence ∇F is a vector perpendicular to the surface $F =$ const. Now ∇F may be a vector of the (δ)-class or of the (γ)-class, and the tangent plane is then respectively a (γ)-plane or a (δ)-plane.[29]

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[29] In our non-Euclidean geometry ∇F will not be a vector in the line of the greatest change of F. If $d\mathbf{r}$ be written as $\mathbf{u}\,ds$, where $\mathbf{u}$ is a unit vector in the

If we select three mutually perpendicular axes $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_4$, and denote by x_1, x_2, x_4 the coordinates (intervals) along these axes, then

$$dF = dx_1 \frac{\partial F}{\partial x_1} + dx_2 \frac{\partial F}{\partial x_2} + dx_4 \frac{\partial F}{\partial x_4} = (dx_1\mathbf{k}_1 + dx_2\mathbf{k}_2 + dx_4\mathbf{k}_4)\cdot\nabla F.$$

From this ∇ may be determined as

$$\nabla = \mathbf{k}_1 \frac{\partial}{\partial x_1} + \mathbf{k}_2 \frac{\partial}{\partial x_2} - \mathbf{k}_4 \frac{\partial}{\partial x_4}\cdot \qquad (40)$$

Thus ∇ appears formally as a 1-vector, and may be treated formally as such.[30]

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direction of $d\mathbf{r}$ and where ds is the interval or magnitude of $d\mathbf{r}$, we may write

$$dF = ds\,\mathbf{u}\cdot\nabla F \qquad \text{or} \qquad \mathbf{u}\cdot\nabla F = \frac{dF}{ds}\cdot$$

Hence the component of ∇F along the direction $\mathbf{dr}$ is the directional derivative of F in that direction. Consider now two neighboring surfaces of constant F. Suppose first that the (approximately parallel) tangent planes to the surfaces are of class (γ), so that the perpendicular ∇F is a (δ)-vector. Then, in our geometry, the perpendicular from a point on one surface to a point of the other is, of all lines of its class, the line of greatest interval ds (§12). The directional derivative along the normal is therefore numerically a minimum (instead of a maximum) relative to neighboring directions. In fact, the derivative along a line of fixed direction would be infinite, because along the fixed cone $ds = 0$. Along the (γ)-lines the directional derivative varies between 0 and ∞. Suppose next that the tangent planes are of class (δ), so that the perpendicular ∇F is a (γ)-line. Then the interval ds along the perpendicular from a point on one surface to a point on the other is neither a maximum nor a minimum, but a minimax. For it is less than along any neighboring direction (of the same class) which with the perpendicular determines a (γ)-plane, but greater than along any neighboring direction (of the same class) which with the perpendicular determines a (δ)-plane.

[30] The above definition of ∇F depends on inner multiplication, and hence upon the notion of perpendicularity or rotation. It is, however, interesting to observe that we may define a differential operator ∇' dependent upon the outer product, and hence upon the idea of translation alone. The definition would then read

$$\mathbf{a}\times\mathbf{b}\times\mathbf{c}\,dF = d\mathbf{r}\times\nabla' F = (\mathbf{a}dx_1 + \mathbf{b}dx_2 + \mathbf{c}dx_3)\times\nabla' F,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are any three independent vectors, and where x_1, x_2, x_3 are coordinates referred to a set of axes along $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Then

$$\nabla' = \mathbf{b}\times\mathbf{c}\frac{\partial}{\partial x_1} + \mathbf{c}\times\mathbf{a}\frac{\partial}{\partial x_2} + \mathbf{a}\times\mathbf{b}\frac{\partial}{\partial x_3}\cdot \qquad (41)$$

Now ∇' may be regarded as a 2-vector operator in the same sense as ∇ is regarded as a 1-vector. To show the relation of ∇' to ∇, when the ideas of perpendicularity are assumed, we may take $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_4$ and x_3 as x_4. Then

$$\nabla' = \mathbf{k}_{24}\frac{\partial}{\partial x_1} + \mathbf{k}_{41}\frac{\partial}{\partial x_2} + \mathbf{k}_{12}\frac{\partial}{\partial x_4} = \left(\mathbf{k}_1\frac{\partial}{\partial x_1} + \mathbf{k}_2\frac{\partial}{\partial x_2} - \mathbf{k}_4\frac{\partial}{\partial x_4}\right)^*\cdot$$

Thus ∇' is the complement ∇^* of ∇. In fact if

$$(dF)^* = d\mathbf{r}\times\nabla' F \qquad \text{and} \qquad dF = d\mathbf{r}\cdot\nabla F,$$

our rule of operation (30) shows that $\nabla' = \nabla^*$.

If we consider a field of 1-vectors, that is, a 1-vector function **f** of position in space, we are naturally led to enquire what meaning, if any, should be associated with the formal combinations

$$\nabla \cdot \mathbf{f} \quad \text{and} \quad \nabla \times \mathbf{f}$$

obtained by operating with the 1-vector ∇. Let

$$\mathbf{f}(x_1, x_2, x_4) = f_1\mathbf{k}_1 + f_2\mathbf{k}_2 + f_4\mathbf{k}_4.$$

Then

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_4}{\partial x_4},$$

$$\nabla \times \mathbf{f} = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\mathbf{k}_{12} + \left(\frac{\partial f_4}{\partial x_1} + \frac{\partial f_1}{\partial x_4}\right)\mathbf{k}_{14} + \left(\frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4}\right)\mathbf{k}_{24}.$$

Of these the first, $\nabla \cdot \mathbf{f}$, is a scalar function of position, and the second, $\nabla \times \mathbf{f}$, is a 2-vector function of position. They correspond respectively to the divergence and curl in Euclidean three dimensional space. The first, $\nabla \cdot \mathbf{f}$, has indeed the same form as usual. And this was to be expected: for physically or geometrically the idea of divergence depends on translation alone and not on rotation, and it would also have appeared analytically evident if we had used in the definition of divergence the operator ∇^* instead of ∇. The second, $\nabla \times \mathbf{f}$, differs from the ordinary curl not only in that we have retained it as a 2-vector (instead of replacing it by the 1-vector, its complement, as is usually done in Euclidean geometry of three dimensions), but also in that it represents non-Euclidean rotation in the vector field in the same sense that the curl represents ordinary rotation.

If F is a scalar function of position, then ∇F is a 1-vector function. We may then form

$$\nabla \cdot \nabla F, \quad \nabla \times \nabla F.$$

Of these the second, $\nabla \times \nabla F$, vanishes identically, as may be seen by its expansions or by regarding it as an outer product in which one vector is repeated. The first, $\nabla \cdot \nabla F$, may be expanded as

$$\nabla \cdot \nabla F = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} - \frac{\partial^2 F}{\partial x_4^2},$$

and $\nabla \cdot \nabla$ corresponds to Laplace's operator in Euclidean geometry.

If **f** is a 1-vector function, there are four different expressions which involve the operator ∇ twice, namely

$$\nabla\nabla \cdot \mathbf{f}, \qquad \nabla \cdot \nabla \mathbf{f}, \qquad \nabla \cdot \nabla \times \mathbf{f}, \qquad \nabla \times \nabla \times \mathbf{f}.$$

Of these the last is a 3-vector function, which clearly vanishes identically. The first three are 1-vector functions, and are connected by the relation

$$\nabla \cdot \nabla \times \mathbf{f} = \nabla (\nabla \cdot \mathbf{f}) - \nabla \cdot \nabla \mathbf{f},$$

as may be seen by expansion or by the application of (34).

Kinematics and Dynamics in a Plane.

35. The three dimensional non-Euclidean geometry which we have developed is adapted to the discussion of the kinematics and dynamics of a particle constrained to move in a plane. The two dimensions of space and the one of time constitute the three dimensions of our manifold. Any (γ)-plane in this manifold may be called space, and extension along the complementary (δ)-line may be called time. As in the simpler case, any (δ)-line represents the locus in time and space of an unaccelerated particle, and any (δ)-curve the locus of an accelerated particle. If we choose any two perpendicular axes x_1, x_2 of space, and the perpendicular time axis x_4, then if the locus of any particle is inclined at the non-Euclidean angle ϕ to the chosen time axis, the particle is said to be in motion with the velocity **v** of which the magnitude is $v = \tanh \phi$.

For the locus of a particle let

$$s = \int \sqrt{dx_4{}^2 - dx_1{}^2 - dx_2{}^2}$$

be the arc measured along the (δ)-curve, and let **r** be the radius vector from any origin to a point of the curve. Then the derivative of **r** by s is the unit tangent **w** to the curve. We have

$$\mathbf{w} = \mathbf{k}_1 \frac{dx_1}{ds} + \mathbf{k}_2 \frac{dx_2}{ds} + \mathbf{k}_4 \frac{dx_4}{ds}$$

If the velocity **v** is $$\mathbf{v} = \mathbf{k}_1 \frac{dx_1}{dx_4} + \mathbf{k}_2 \frac{dx_2}{dx_4},$$

then since $$\frac{dx_4}{ds} = \cosh \phi = \frac{1}{\sqrt{1 - v^2}},$$

we write [31]

$$\mathbf{w} = \frac{1}{\sqrt{1 - v^2}} \left(\mathbf{k}_1 \frac{dx_1}{dx_4} + \mathbf{k}_2 \frac{dx_2}{dx_4} + \mathbf{k}_4 \right) = \frac{\mathbf{v} + \mathbf{k}_4}{\sqrt{1 - v^2}}. \qquad (42)$$

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[31] By a transformation to a new set of axes we may derive at once the general form of Einstein's equation for the addition of velocities.

To obtain the vector curvature of the locus we write

$$\mathbf{c} = \frac{d\mathbf{w}}{ds} = \frac{dx_4}{ds}\frac{d\mathbf{w}}{dx_4} = \frac{1}{1 - v^2}\frac{d\mathbf{v}}{dx_4} + \frac{\mathbf{v} + \mathbf{k}_4}{(1 - v^2)^2}\, v \frac{dv}{dx_4},$$

or

$$\mathbf{c} = \frac{1}{1 - v^2}\frac{d\mathbf{v}}{dt} + \frac{\mathbf{v} + \mathbf{k}_4}{(1 - v^2)^2}\, v \frac{dv}{dt}. \tag{43}$$

If $\mathbf{v}$ be written as $\mathbf{v} = v\mathbf{u}$, where $\mathbf{u}$ is a unit vector, the resolution of $\mathbf{c}$ into three mutually perpendicular components along $\mathbf{u}$, $\mathbf{k}_4$, and $d\mathbf{u}$ follows immediately:

$$\mathbf{c} = \frac{\mathbf{u}\dfrac{dv}{dt}}{(1 - v^2)^2} + \frac{v\dfrac{d\mathbf{u}}{dt}}{1 - v^2} + \frac{v\mathbf{k}_4\dfrac{dv}{dt}}{(1 - v^2)^2}. \tag{44}$$

The magnitude of $\mathbf{c}$ is

$$\begin{aligned}\sqrt{\mathbf{c}\cdot\mathbf{c}} &= \left[\frac{\left(\dfrac{dv}{dt}\right)^2}{(1 - v^2)^3} + \frac{v^2\dfrac{d\mathbf{u}}{dt}\cdot\dfrac{d\mathbf{u}}{dt}}{(1 - v^2)^2}\right]^{\frac{1}{2}} \\ &= \frac{1}{1 - v^2}\left[\dot{\mathbf{v}}\cdot\dot{\mathbf{v}} + \frac{1}{1 - v^2}(\mathbf{v}\cdot\dot{\mathbf{v}})^2\right]^{\frac{1}{2}} \\ &= \frac{1}{(1 - v^2)^{\frac{3}{2}}}\left[\dot{\mathbf{v}}\cdot\dot{\mathbf{v}} - (\mathbf{v}\times\dot{\mathbf{v}})\cdot(\mathbf{v}\times\dot{\mathbf{v}})\right]^{\frac{1}{2}}.\end{aligned} \tag{45}$$

In case the acceleration is along the line of motion, these expressions reduce to those previously found; the additional term is due to the acceleration normal to the line of motion.

36. Mass may now be introduced just as in the simpler case already discussed, and here likewise we are led to the equation

$$m = \frac{m_0}{\sqrt{1 - v^2}}.$$

The extended momentum in this case is also $m_0\mathbf{w}$, that is,

$$m_0\mathbf{w} = m\mathbf{v} + m\mathbf{k}_4. \tag{46}$$

We may speak of $\mathbf{w}$ as the extended velocity, of $\mathbf{c}$ as the extended acceleration, and of $m_0\mathbf{c}$ as the extended force. It is to be noted that while ordinary momentum is the space component of extended momentum, ordinary velocity, acceleration, and force are not the space components of the corresponding extended vectors. Indeed the space component of the extended velocity is $\mathbf{v}/\sqrt{1-v^2}$. The ordinary force, measured as rate of change of momentum, is

$$\mathbf{f} = \frac{dm\mathbf{v}}{dt} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} = \frac{m_0\mathbf{u}\dfrac{dv}{dt}}{(1-v^2)^{\frac{3}{2}}} + \frac{m_0 v\dfrac{d\mathbf{u}}{dt}}{(1-v^2)^{\frac{1}{2}}}, \tag{47}$$

which is the space component of $m_0\mathbf{c}$ multiplied by $\sqrt{1-v^2}$.

It is evident that in our mechanics the equations

$$\mathbf{f} = \frac{dm\mathbf{v}}{dt} \quad \text{and} \quad \mathbf{f} = m\mathbf{a},$$

where $\mathbf{a} = d\mathbf{v}/dt$, are not equivalent, and it is the first of these which we have chosen as fundamental. This makes the mass a definite scalar property of the system. Those who have used the second of the equations have been led to the idea of a mass which is different in different directions, and indeed have introduced as the "longitudinal" and the "transverse" mass the coefficients

$$\frac{m_0}{(1-v^2)^{\frac{3}{2}}}, \quad \frac{m_0}{(1-v^2)^{\frac{1}{2}}}$$

of the components of acceleration along the path and perpendicular to it, that is, of the longitudinal and transverse accelerations, which are respectively

$$\mathbf{u}\frac{dv}{dt}, \quad v\frac{d\mathbf{u}}{dt}.$$

The disadvantages of this latter system are obvious.

An interesting case of planar motion is that under a force constant in magnitude and in direction, say $f_x = 0$, $f_y = -k$. The momentum in the x-direction is constant, that in the y-direction is equal to its initial value less kt. From these two equations the integration may be completed. Or, in place of the second, the fact that the increase in mass (that is, energy) is equal to the work done by the force, may be used to give a second equation. The trajectory of the particle is not a parabola, but a curve of the form $y + a = -b\cosh(cx - d)$, resembling a catenary.

The space-time locus of uniform circular motion is a helix

$$\mathbf{r} = a(\mathbf{k}_1\cos nt + \mathbf{k}_2\sin nt) + \mathbf{k}_4 t.$$

Then

$$m\mathbf{v} = man(-\mathbf{k}_1 \sin nt + \mathbf{k}_2 \cos nt) + m\mathbf{k}_4,$$

$$\mathbf{f} = \frac{dm\mathbf{v}}{dt} = -man^2(\mathbf{k}_1 \cos nt + \mathbf{k}_2 \sin nt) = -mn^2\mathbf{r}_s,$$

where $\mathbf{r}_s$ is the component of $\mathbf{r}$ on the two-dimensional "space." The force is directed toward the center, as usual. It may be observed that if in general the force is central, the moment of momentum is constant. For if

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{f}, \qquad \mathbf{r}_s \times \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(\mathbf{r}_s \times m\mathbf{v}) = \mathbf{r}_s \times \mathbf{f} = 0.$$

That the rate of change of moment of momentum is equal to the moment of the force is therefore a principle which holds in non-Newtonian as in ordinary mechanics.

THE NON-EUCLIDEAN GEOMETRY IN FOUR DIMENSIONS.

Geometry and Vector Algebra.

37. Consider now a space of four dimensions in which the elements are points, lines, planes, flat 3-spaces or planoids, and which is subject to the same rules of translation or parallel-transformation as two or three dimensional space. If **a** and **b** are two 1-vectors, the product $\mathbf{a}\times\mathbf{b}$ is a 2-vector, that is, the parallelogram determined by the vectors. The parallelograms $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ will be taken as of opposite sign, but otherwise equal. The equation $\mathbf{a}\times\mathbf{b} = 0$ expresses the condition that **a** and **b** are parallel. If **c** is any third 1-vector, not lying in the plane of **a** and **b**, the product $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$, which is now itself a vector will represent the parallelepiped determined by the three vectors. The condition $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = 0$ therefore states that the three 1-vectors lie in a plane. If now **d** is a fourth 1-vector, not lying in the 3-space or planoid determined by **a**, **b**, **c**, the product $\mathbf{a}\times\mathbf{b}\times\mathbf{c}\times\mathbf{d}$ will represent the four dimensional parallel figure determined by the vectors. This product is a pseudo-scalar of which the magnitude is the four dimensional content of the parallel figure. The condition $\mathbf{a}\times\mathbf{b}\times\mathbf{c}\times\mathbf{d} = 0$ shows that the four vectors lie in some planoid. In all these outer products the sign is changed by the interchange of two adjacent factors, as in the case of lower dimensions. Moreover, the associative law, the distributive law, and the law of association for scalar factors will also hold, as is evident from their geometrical interpretation.

Two 1-vectors are added in the ordinary way by the parallelogram law. The same is true of two 2-vectors if they intersect in a line, that is, if they lie in the same 3-space (§ 25). It is, however, clear that in four dimensional space it is possible to have two parallelograms which have a common vertex but which do not lie in any planoid, that is, do not intersect in a line. For two such 2-vectors the construction previously given for the sum is not applicable, and it is indeed impossible to replace the sum of the two 2-vectors by a single plane vector. The sum may, however, be replaced in an infinite variety of ways by the sum of two other 2-vectors. For if **A** and **B** are any two 2-vectors, and if **a** and **b** be two 1-vectors drawn respectively in the planes of **A** and **B**, then the 2-vector $\mathbf{a}\times\mathbf{b} = \mathbf{C}$ may be added or subtracted from **A** and **B** so that

$$\mathbf{A} + \mathbf{B} = (\mathbf{A} + \mathbf{C}) + (\mathbf{B} - \mathbf{C}) = \mathbf{A}' + \mathbf{B}'.$$

The sum of more than two 2-vectors can, however, always be reduced to a sum of two. For if three planes in four dimensional space pass through a point, at least two must intersect in a line. A sum of 2-vectors, which is not reducible to a single uniplanar or simple 2-vector will be called a biplanar or double 2-vector whenever it is important to emphasize the difference. Since the analytical treatment of these two kinds of 2-vectors is not materially different, they will be designated by the same type of letters (clarendon capitals).

A vector of the type $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$ will be called a 3-vector. As two planoids which have a point in common, intersect in a plane, a geometric construction for the sum of two 3-vectors may be given in a manner which is the immediate extension of the rule for 2-vectors in three dimensional space. The sum of two 3-vectors is always a simple 3-vector.

In respect to rotation and to the classification of lines, planes, and planoids, our four dimensional geometry will be non-Euclidean in such a manner as to be the natural extension of the non-Euclidean geometries of two and three dimensions which have been already discussed. As in two dimensions there were two fixed lines through a point, and in three dimensions a fixed cone, so in four dimensions there will be a fixed conical spread of three dimensions, or hypercone, which separates lines within the hypercone and called (δ)-lines, from lines outside the hypercone, which are called (γ)-lines. Besides the singular planes which are tangent to the hypercone, there are two classes of planes, namely, (δ)-planes which contain a (δ)-line, and (γ)-planes which contain no (δ)-line. Besides the singular planoids which are tangent to the hypercone, there are two classes of planoids, namely, (δ)-planoids which contain a (δ)-line, and (γ)-planoids which contain no (δ)-line. In the (γ)-planoids the geometry is the ordinary three dimensional Euclidean geometry; in the (δ)-planoids the geometry is that three dimensional non-Euclidean geometry which we have discussed at length.

Every (δ)-line determines a perpendicular planoid of class (γ), and every (γ)-line determines a perpendicular planoid of class (δ). Thus if we construct four mutually perpendicular lines, one will be a (δ)-line, and three will be (γ)-lines. A plane determined by one pair of these four mutually perpendicular lines is *completely* perpendicular to the plane determined by the other pair, in the sense that every line of one plane is perpendicular to every line of the other, and the planes therefore have no line in common. In general every plane determines uniquely a completely perpendicular plane. One of the planes is a (γ)-plane and the other is a (δ)-plane.

As in our previous geometries, perpendiculars remain perpendicular during rotation. If then in a rotation any plane remains fixed, its completely perpendicular plane will also remain fixed; and a general rotation may be regarded as the combination of a certain ordinary Euclidean rotation in a certain (γ)-plane, combined with a certain non-Euclidean rotation in the completely perpendicular (δ)-plane.

38. Let $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ be four mutually perpendicular unit vectors of which the last is a (δ)-vector. The six coordinate 2-vectors may then be designated [32] as $\mathbf{k}_{14}$, $\mathbf{k}_{24}$, $\mathbf{k}_{34}$, $\mathbf{k}_{23}$, $\mathbf{k}_{31}$, $\mathbf{k}_{12}$. There are furthermore four coordinate unit 3-vectors $\mathbf{k}_{234}$, $\mathbf{k}_{314}$, $\mathbf{k}_{124}$, $\mathbf{k}_{123}$; and a unit pseudo-scalar $\mathbf{k}_{1234}$. We may represent 1-vectors, 2-vectors and 3-vectors, as the sum of their projections on the coordinate axes, coordinate planes, and coordinate planoids. Thus

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_3\mathbf{k}_3 + a_4\mathbf{k}_4,$$

$$\mathbf{A} = A_{14}\mathbf{k}_{14} + A_{24}\mathbf{k}_{24} + A_{34}\mathbf{k}_{34} + A_{23}\mathbf{k}_{23} + A_{31}\mathbf{k}_{31} + A_{12}\mathbf{k}_{12},$$

$$\boldsymbol{\mathfrak{A}} = \mathfrak{A}_{234}\mathbf{k}_{234} + \mathfrak{A}_{314}\mathbf{k}_{314} + \mathfrak{A}_{124}\mathbf{k}_{124} + \mathfrak{A}_{123}\mathbf{k}_{123}.$$

The outer product of any two vectors is defined geometrically and expressed analytically in a manner entirely analogous to that of the simpler cases already discussed. We thus obtain the following equations for the different types of products.

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[32] The particular order of subscripts is chosen for convenience only.

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a} = {} & (a_1b_4 - a_4b_1)\mathbf{k}_{14} + (a_2b_4 - a_4b_2)\mathbf{k}_{24} + (a_3b_4 - a_4b_3)\mathbf{k}_{34} \\ & + (a_2b_3 - a_3b_2)\mathbf{k}_{23} + (a_3b_1 - a_1b_3)\mathbf{k}_{31} + (a_1b_2 - a_2b_1)\mathbf{k}_{12},\end{aligned}$$

$$\begin{aligned}\mathbf{a}\times\mathbf{A} = {} & (a_2A_{34} - a_3A_{24} + a_4A_{23})\mathbf{k}_{234} + (a_3A_{14} - a_1A_{34} + a_4A_{31})\,\mathbf{k}_{314} \\ & + (a_1A_{24} - a_2A_{14} + a_4A_{12})\,\mathbf{k}_{124} + (a_1A_{23} + a_2A_{31} + a_3A_{12})\,\mathbf{k}_{123},\end{aligned}$$

$$\mathbf{a}\times\boldsymbol{\mathfrak{A}} = -\,\boldsymbol{\mathfrak{A}}\times\mathbf{a} = (a_1\mathfrak{A}_{234} + a_2\mathfrak{A}_{314} + a_3\mathfrak{A}_{124} - a_4\mathfrak{A}_{123})\mathbf{k}_{1234},$$

$$\mathbf{A}\times\mathbf{B} = (A_{14}B_{23} + A_{24}B_{31} + A_{34}B_{12} + A_{23}B_{14} + A_{31}B_{24} + A_{12}B_{34})\,\mathbf{k}_{1234}.$$

The outer product of two vectors the sum of whose dimensions is greater than four vanishes. The outer product of a vector by itself vanishes except in the case of the biplanar or double 2-vector where the product becomes

$$\mathbf{A}\times\mathbf{A} = 2(A_{14}A_{23} + A_{24}A_{31} + A_{34}A_{12})\,\mathbf{k}_{1234}.$$

If the biplanar vector be written as $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where **B** and **C** are two simple plane vectors, the product may be written

$$\mathbf{A}\times\mathbf{A} = (\mathbf{B} + \mathbf{C})\times(\mathbf{B} + \mathbf{C}) = 2\mathbf{B}\times\mathbf{C}.$$

It thus appears that $\mathbf{A}\times\mathbf{A}$ is twice the four dimensional parallelepiped constructed upon any pair of planes into which the double vector may be resolved. The vanishing of the outer product, $\mathbf{A}\times\mathbf{A} = 0$, is the necessary and sufficient condition that **A** be uniplanar.

The general rule for all cases of inner product has been stated (§ 29). We may tabulate the following cases.

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4,$$

$$\begin{aligned}\mathbf{a}\cdot\mathbf{A} = {} & (a_2A_{12} - a_3A_{31} - a_4A_{14})\,\mathbf{k}_1 + (-\,a_1A_{12} + a_3A_{23} - a_4A_{24})\,\mathbf{k}_2 \\ & + (a_1A_{31} - a_2A_{23} - a_4A_{34})\,\mathbf{k}_3 + (-\,a_1A_{14} - a_2A_{24} - a_3A_{34})\,\mathbf{k}_4,\end{aligned}$$

$$\begin{aligned}\mathbf{a}\cdot\boldsymbol{\mathfrak{A}} = {} & (a_3\mathfrak{A}_{314} - a_2\mathfrak{A}_{124})\,\mathbf{k}_{14} + (a_1\mathfrak{A}_{124} - a_3\mathfrak{A}_{234})\,\mathbf{k}_{24} \\ & + (a_2\mathfrak{A}_{234} - a_1\mathfrak{A}_{314})\,\mathbf{k}_{34} + (a_1\mathfrak{A}_{123} - a_4\mathfrak{A}_{234})\,\mathbf{k}_{23} \\ & + (a_2\mathfrak{A}_{123} - a_4\mathfrak{A}_{314})\,\mathbf{k}_{31} + (a_3\mathfrak{A}_{123} - a_4\mathfrak{A}_{124})\,\mathbf{k}_{12},\end{aligned}$$

$$\mathbf{A}\cdot\mathbf{B} = -\,A_{14}B_{14} - A_{24}B_{24} - A_{34}B_{34} + A_{23}B_{23} + A_{31}B_{31} + A_{12}B_{12},$$

$$\begin{aligned}\mathbf{A}\cdot\boldsymbol{\mathfrak{A}} = {} & (-\,A_{24}\mathfrak{A}_{124} + A_{34}\mathfrak{A}_{314} + A_{23}\mathfrak{A}_{123})\,\mathbf{k}_1 + (A_{14}\mathfrak{A}_{124} - A_{34}\mathfrak{A}_{234} \\ & + A_{31}\mathfrak{A}_{123})\,\mathbf{k}_2 + (-\,A_{14}\mathfrak{A}_{314} + A_{24}\mathfrak{A}_{234} + A_{12}\mathfrak{A}_{123})\,\mathbf{k}_3 \\ & + (A_{23}\mathfrak{A}_{234} + A_{31}\mathfrak{A}_{314} + A_{12}\mathfrak{A}_{124})\,\mathbf{k}_4,\end{aligned}$$

$$\boldsymbol{\mathfrak{A}}\cdot\boldsymbol{\mathfrak{B}} = -\,\mathfrak{A}_{234}\mathfrak{B}_{234} - \mathfrak{A}_{314}\mathfrak{B}_{314} - \mathfrak{A}_{124}\mathfrak{B}_{124} + \mathfrak{A}_{123}\mathfrak{B}_{123}.$$

The geometrical interpretation of these inner products follows the same lines as before. The inner product of a vector into a vector of equal dimensions is a scalar, and is the product of either into the projection of the other upon it. In the case where a biplanar 2-vector is projected, or is projected upon, each simple plane has to be treated, and the results compounded. That this may be done follows at once from the distributive law. The product of two vectors of different dimensionality is a vector of which the dimension is the difference of the dimensions of the factors; this vector lies in the factor of larger dimensions and is perpendicular to the factor of smaller dimensions. However, the product $\mathbf{a}\cdot\mathbf{A}$, if $\mathbf{A}$ is biplanar, is compounded of two 1-vectors lying in the two component planes.

The complement of a vector is again defined as its inner product with the unit pseudo-scalar $\mathbf{k}_{1234}$. The complement of a 1-vector is a perpendicular 3-vector, and vice-versa; that of a simple 2-vector is the completely perpendicular 2-vector. We may tabulate the results for the unit vectors.

$$\mathbf{k}_1{}^* = -\mathbf{k}_{234}, \quad \mathbf{k}_2{}^* = -\mathbf{k}_{314}, \quad \mathbf{k}_3{}^* = -\mathbf{k}_{124}, \quad \mathbf{k}_4{}^* = -\mathbf{k}_{123},$$

$$\mathbf{k}_{14}{}^* = -\mathbf{k}_{23}, \quad \mathbf{k}_{24}{}^* = -\mathbf{k}_{31}, \quad \mathbf{k}_{34}{}^* = -\mathbf{k}_{12},$$

$$\mathbf{k}_{23}{}^* = \mathbf{k}_{14}, \quad \mathbf{k}_{31}{}^* = \mathbf{k}_{24}, \quad \mathbf{k}_{12}{}^* = \mathbf{k}_{34},$$

$$\mathbf{k}_{234}{}^* = -\mathbf{k}_1, \quad \mathbf{k}_{314}{}^* = -\mathbf{k}_2, \quad \mathbf{k}_{124}{}^* = -\mathbf{k}_3, \quad \mathbf{k}_{123}{}^* = -\mathbf{k}_4.$$

With the aid of complements a unique resolution of a given 2-vector into two completely perpendicular parts may be accomplished. Suppose the resolution effected as

$$\mathbf{A} = m\mathbf{M} + n\mathbf{N}$$

where $\mathbf{M}$ is a unit vector of class (γ) and $\mathbf{N}$ one of class (δ) so chosen that $\mathbf{M}\times\mathbf{N}$ is a positive unit pseudo-scalar. Then

$$\mathbf{A}^* = -n\mathbf{M} + m\mathbf{N},$$

and $$\mathbf{M} = \frac{m\mathbf{A} - n\mathbf{A}^*}{m^2 + n^2}, \qquad \mathbf{N} = \frac{n\mathbf{A} + m\mathbf{A}^*}{m^2 + n^2}.$$

Hence $$\mathbf{A} = \frac{m^2\mathbf{A} - mn\mathbf{A}^*}{m^2 + n^2} + \frac{n^2\mathbf{A} + nm\mathbf{A}^*}{m^2 + n^2}.$$

Let $$p = \mathbf{A}\cdot\mathbf{A} = m^2 - n^2, \qquad q = \mathbf{A}\cdot\mathbf{A}^* = -2mn.$$

The quantities m, n may then be expressed in terms of p, q, that is, in terms of $\mathbf{A}\cdot\mathbf{A}$, $\mathbf{A}\cdot\mathbf{A}^*$. The result is

$$\mathbf{A} = \tfrac{1}{2}\frac{(\sqrt{p^2 + q^2} + p)\,\mathbf{A} + q\mathbf{A}^*}{\sqrt{p^2 + q^2}} + \tfrac{1}{2}\frac{(\sqrt{p^2 + q^2} - p)\,\mathbf{A} - q\mathbf{A}^*}{\sqrt{p^2 + q^2}}.$$

The general relationships between products of vectors and their complements have been developed in a previous section for a space of any dimensions. It was there shown that (except 37) formulas (34)–(39) for the expansion of all types of products involving 1-vectors and 2-vectors would be true in higher dimensions, and this is true even if the 2-vectors involved happen to be biplanar, because any such vectors is the sum of two uniplanar vectors and the equations are linear or bilinear in the vectors. Similar equations may, if occasion requires, be developed for products involving 3-vectors.

39. We have not yet considered those vectors whose inner products with themselves are zero. The case of the 1-vector, which is an element of the hypercone, need not be treated again in detail. For such a vector

$$\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 - a_4^2 = 0.$$

A uniplanar 2-vector such that $\mathbf{A}\cdot\mathbf{A} = 0$ satisfies the conditions

$$\mathbf{A}\times\mathbf{A} = 2\,(A_{14}A_{23} + A_{24}A_{31} + A_{34}A_{12})\,\mathbf{k}_{1234} = 0,$$

$$\mathbf{A}\cdot\mathbf{A} = -A_{14}^2 - A_{24}^2 - A_{34}^2 + A_{23}^2 + A_{31}^2 + A_{12}^2 = 0.$$

Such a vector is obviously a plane tangent to the hypercone; for it can be neither a (γ)- nor a (δ)-plane. The singular plane has the same properties as in three dimensional space. The element of tangency may be found as follows. If $\mathbf{a}$ is any vector, $\mathbf{a}\cdot\mathbf{A}$ is a line in the plane $\mathbf{A}$, and $(\mathbf{a}\cdot\mathbf{A})\cdot\mathbf{A}$ is a perpendicular line of the plane. But the only line which is perpendicular to another line in this peculiar two dimensional space is the singular line, that is, the element of tangency with the hypercone. If $\mathbf{k}_4$ be taken as $\mathbf{a}$, the element may be written as

$$\begin{aligned}(\mathbf{k}_4\cdot\mathbf{A})\cdot\mathbf{A} = {} & \mathbf{k}_1\,(A_{34}A_{31} - A_{24}A_{12}) + \mathbf{k}_2\,(A_{14}A_{12} - A_{34}A_{23}) \\ & + \mathbf{k}_3\,(A_{24}A_{23} - A_{14}A_{31}) + \mathbf{k}_4\,(A_{14}^2 + A_{24}^2 + A_{34}^2),\end{aligned}$$

an equation which we shall find serviceable.

The complement of a uniplanar singular 2-vector is itself such a vector, and it may readily be shown to pass through the same element of tangency. Indeed through every element of the hypercone is a whole single infinity of tangent planes which are mutually complementary in pairs.

If a 2-vector be biplanar, that is, if $\mathbf{A}\times\mathbf{A}$ is not zero, the condition $\mathbf{A}\cdot\mathbf{A} = 0$ is satisfied when, if the vector be resolved into the two complementary (γ)- and (δ)-vectors, these have the same magnitude. For if

$$\mathbf{A} = m\mathbf{M} + n\mathbf{N}, \qquad \mathbf{A}\cdot\mathbf{A} = m^2 - n^2.$$

Such a vector is singular only in an analytical sense.

The complement of a singular 1-vector is a 3-vector which itself is evidently singular. It is the planoid tangent to the hypercone through the given element.[33] It contains, besides the pencil of singular planes through the element of tangency, only (γ)-planes.

We may take this opportunity of summarizing the properties of singular vectors in general. The inner product of any singular vector by itself is 0. Every singular vector is perpendicular to itself and to every singular vector lying within it. The magnitude of a singular vector is zero. This does not imply that such a vector is not a definite geometric object, but only that the interval of a singular 1-vector, the area of a singular 2-vector, and the volume of a singular 3-vector are zero when compared with non-singular intervals, areas, and volumes.

The visualization of the geometrical properties of a four dimensional and especially of a non-Euclidean four dimensional geometry is extremely difficult. It is of course possible to rely wholly on the analytic relations, and thus avoid the difficulty. But we believe that it is of the greatest importance to realize that we are dealing with perfectly definite geometrical objects which are independent of any arbitrary axes of reference, and that it is therefore advisable to make every effort toward the visualization. It seems probable that Minkowski, although he employed chiefly the analytical point of view in his great memoir, must himself have largely employed the geometrical method in his thinking.

The Differentiating Operator ◇.

40. By analogy we may in four dimensions define the operator ◇, called quad, by the equation

$$d(\ ) = d\mathbf{r}\cdot\Diamond(\ ). \tag{48}$$

When referred to a set of perpendicular axes, quad takes the form

$$\Diamond = \mathbf{k}_1\frac{\partial}{\partial x_1} + \mathbf{k}_2\frac{\partial}{\partial x_2} + \mathbf{k}_3\frac{\partial}{\partial x_3} - \mathbf{k}_4\frac{\partial}{\partial x_4}, \tag{49}$$

and like ∇ it may be regarded formally as a 1-vector.

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[33] The geometry in a singular planoid is analogous to that in a singular plane (§ 31). In this 3-space there are two classes of lines, singular lines, all of which are parallel to each other, and non-singular lines, (γ)-lines, all of which are perpendicular to the singular lines. Similarly there are two classes of planes, singular planes, all of which are parallel to the singular lines, and non-singular (γ)-planes, which are perpendicular to every singular plane. Volumes are comparable with one another but are all of zero magnitude as compared with a volume in any non-singular planoid.

We may therefore write the following equations. The result of applying $\Diamond$ to a scalar function F is a 1-vector $\Diamond F$, which might be called the gradient of F.

$$\Diamond F = \mathbf{k}_1 \frac{\partial F}{\partial x_1} + \mathbf{k}_2 \frac{\partial F}{\partial x_2} + \mathbf{k}_3 \frac{\partial F}{\partial x_3} - \mathbf{k}_4 \frac{\partial F}{\partial x_4}.$$

The application of $\Diamond$ to a 1-vector function $\mathbf{f}$ by inner multiplication is a scalar, which might be called the divergence of $\mathbf{f}$.

$$\Diamond \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4}.$$

The application of $\Diamond$, by outer multiplication, to the 1-vector $\mathbf{f}$ is a 2-vector function, which might be called the curl of $\mathbf{f}$.

$$\begin{aligned}\Diamond \times \mathbf{f} = {} & \left(\frac{\partial f_4}{\partial x_1} + \frac{\partial f_1}{\partial x_4}\right)\mathbf{k}_{14} + \left(\frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4}\right)\mathbf{k}_{24} + \left(\frac{\partial f_4}{\partial x_3} + \frac{\partial f_3}{\partial x_4}\right)\mathbf{k}_{34} \\ & + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\mathbf{k}_{23} + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\mathbf{k}_{31} + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\mathbf{k}_{12}.\end{aligned}$$

The expression $\Diamond \cdot \mathbf{F}$ is a 1-vector.

$$\begin{aligned}\Diamond \cdot \mathbf{F} = {} & \left(\frac{\partial f_{12}}{\partial x_2} - \frac{\partial f_{31}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4}\right)\mathbf{k}_1 + \left(\frac{\partial f_{23}}{\partial x_3} - \frac{\partial f_{12}}{\partial x_1} + \frac{\partial f_{24}}{\partial x_4}\right)\mathbf{k}_2 \\ & + \left(\frac{\partial f_{31}}{\partial x_1} - \frac{\partial f_{23}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_4}\right)\mathbf{k}_3 - \left(\frac{\partial f_{14}}{\partial x_1} + \frac{\partial f_{24}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_3}\right)\mathbf{k}_4.\end{aligned}$$

The product $\Diamond \times \mathbf{F}$ is a 3-vector.

$$\begin{aligned}\Diamond \times \mathbf{F} = {} & \left(\frac{\partial f_{34}}{\partial x_2} - \frac{\partial f_{24}}{\partial x_3} - \frac{\partial f_{23}}{\partial x_4}\right)\mathbf{k}_{234} + \left(\frac{\partial f_{14}}{\partial x_3} - \frac{\partial f_{34}}{\partial x_1} - \frac{\partial f_{31}}{\partial x_4}\right)\mathbf{k}_{314} \\ & + \left(\frac{\partial f_{24}}{\partial x_1} - \frac{\partial f_{14}}{\partial x_2} - \frac{\partial f_{12}}{\partial x_4}\right)\mathbf{k}_{124} + \left(\frac{\partial f_{23}}{\partial x_1} + \frac{\partial f_{31}}{\partial x_2} + \frac{\partial f_{12}}{\partial x_3}\right)\mathbf{k}_{123}.\end{aligned}$$

We might likewise expand $\Diamond \cdot \mathfrak{F}$ and $\Diamond \times \mathfrak{F}$.

The rules (30) and (24) for operation with the complement enable us to write

$$(\Diamond \cdot a)^* = -\Diamond \times a^*, \qquad (\Diamond \times a)^* = \Diamond \cdot a^*,$$

when a is a vector function of any dimensionality in four dimensional space.

It is important to note in all these equations that while quad operates as a 1-vector, it is not a 1-vector in any geometrical sense.

Thus we find, for example, that $\Diamond\times\mathbf{f}$ is not always a plane passing through $\mathbf{f}$, and in fact will usually be a biplanar vector. Also $\Diamond\cdot\mathbf{F}$ is not necessarily in the plane of $\mathbf{F}$.

We have used the same symbol $\Diamond$ for our differential operator as was used by Lewis in his discussion of the vector analysis of four dimensional Euclidean space, and which corresponded to the "lor" of Minkowski. There seems no danger of confusion, since it will never be desirable to work simultaneously in Euclidean and non-Euclidean geometry. Sommerfeld[34] has also developed a vector analysis of essentially Euclidean four dimensional space, and his notation is an extension of that current in Germany for the three dimensional case. For the sake of reference we will compare the two notations, as far as the differential operator is concerned, in the following table.

$$\Diamond F \backsim \text{Grad}\, F,$$
$$\Diamond\cdot\mathbf{f} \backsim \text{Div}\, \mathbf{f},$$
$$\Diamond\times\mathbf{f} \backsim \text{Rot}\, \mathbf{f},$$
$$\Diamond\cdot\mathbf{F} \backsim \mathfrak{Div}\, \mathbf{F}.$$

Operations involving $\Diamond$ twice are of frequent use in a number of important equations. These may be obtained by rules already given if $\Diamond$ be regarded as a 1-vector.

$$\Diamond\times(\Diamond F) = 0, \qquad (50) \qquad\qquad \Diamond\times(\Diamond\times\mathbf{f}) = 0, \qquad (51)$$
$$\Diamond\cdot(\Diamond\cdot\mathbf{F}) = 0, \qquad (52) \qquad\qquad \Diamond\times(\Diamond\times\mathbf{F}) = 0, \qquad (53)$$
$$\Diamond\cdot(\Diamond\cdot\boldsymbol{\mathfrak{F}}) = 0, \qquad (54)$$
$$\Diamond\cdot(\Diamond\times\mathbf{f}) = \Diamond(\Diamond\cdot\mathbf{f}) - (\Diamond\cdot\Diamond)\,\mathbf{f}, \qquad (55)$$
$$\Diamond\cdot(\Diamond\times\mathbf{F}) = \Diamond\times(\Diamond\cdot\mathbf{F}) + (\Diamond\cdot\Diamond)\,\mathbf{F},^{35} \qquad (56)$$
$$\Diamond\cdot(\Diamond\times\boldsymbol{\mathfrak{F}}) = \Diamond\times(\Diamond\cdot\boldsymbol{\mathfrak{F}}) - (\Diamond\cdot\Diamond)\,\boldsymbol{\mathfrak{F}}. \qquad (57)$$

The important operator $\Diamond\cdot\Diamond$ or $\Diamond^2$ has sometimes been called the D'Alembertian. In the expanded form it is

$$\Diamond^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} = \nabla^2 - \frac{\partial^2}{\partial x_4^2}, \qquad (58)$$

where ∇ now denotes the Euclidean differentiating operator in the $\mathbf{k}_{123}$ space.

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[34] Sommerfeld, Ann. d. Physik [4] **33**, 649.

[35] Kraft (Bull. Acad. Cracovie A, 1911, p. 538) devotes a paper to the proof and application of this formula.

41. In the ordinary integral calculus of vectors the theorems due to Gauss and Stokes play an important rôle. In our notation we may express these laws with great simplicity and generalize them to a space of any dimensions. Let us consider first the form of these theorems in the case of two dimensions, beginning with the more familiar Euclidean case.

Stokes's theorem states that the line integral of a vector function **f** around a closed path is equal to the integral of the curl of **f** over the area bounded by the curve. The analytic statement is

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int dS \operatorname{curl} \mathbf{f},$$

where $d\mathbf{s}$ is the vector element of arc, and dS the scalar element of area. In our notation[36] this becomes

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int d\mathbf{S}\cdot\nabla\times\mathbf{f},$$

where $d\mathbf{S}$ is now the 2-vector element of area (a pseudo-scalar) and $\nabla\times\mathbf{f}$ is a pseudo-scalar (the complement of curl **f**, which itself is a scalar in the two dimensional case). Transforming by (35), we may also write

$$\int d\mathbf{s}\cdot\mathbf{f} = -\int\int (d\mathbf{S}\cdot\nabla)\cdot\mathbf{f}. \tag{59}$$

Gauss's theorem states that the integral of the flux of a vector through a closed curve is equal to the integral of the divergence of the vector **f** over the area bounded by the curve. The analytic statement is

$$\int f_n ds = \int\int dS \operatorname{div} \mathbf{f},$$

where f_n is the component of **f** normal to the curve. In our notation this becomes

$$-\int (d\mathbf{s}\times\mathbf{f})^* = \int\int dS\ \nabla\cdot\mathbf{f} = \int\int d\mathbf{S}^*\nabla\cdot\mathbf{f},$$

or, by taking the complement of both sides,

$$-\int d\mathbf{s}\times\mathbf{f} = \int\int d\mathbf{S}\nabla\cdot\mathbf{f};$$

---

[36] One of the advantages of our system of notation is that if one term in an equation is a vector of p dimensions, every other term is a vector of p dimensions. This furnishes at once a check on the correctness of any equation.

and transforming by (36), where in two dimensions $\mathbf{f}\times d\mathbf{S}$ vanishes, we obtain the form

$$\int d\mathbf{s}\times\mathbf{f} = -\int\int (d\mathbf{S}\cdot\nabla)\times\mathbf{f}. \tag{60}$$

Equations (59) and (60) can be combined into the operational equation

$$\int d\mathbf{s}\,(\;) = -\int\int (d\mathbf{S}\cdot\nabla)\,(\;), \tag{61}$$

where the operators may be applied to $\mathbf{f}$ in either inner or outer multiplication.

In three dimensions Stokes's theorem states that the line integral of a vector around a curve is equal to the surface integral of the normal component of the curl of the vector over any surface spanning the curve, with proper regard to sign. The ordinary statement is

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int dS\,(\text{curl }\mathbf{f})_n,$$

which in our notation becomes

$$\int d\mathbf{s}\cdot\mathbf{f} = \int\int d\mathbf{S}\cdot(\nabla\cdot\mathbf{f});$$

and may be transformed by (35) into

$$\int d\mathbf{s}\cdot\mathbf{f} = -\int\int (d\mathbf{S}\cdot\nabla)\cdot\mathbf{f}. \tag{62}$$

In like manner Gauss's theorem states that the integral of the flux of a vector through a closed surface is equal to the integral of the divergence of the vector over the volume inclosed by the surface. Thus, if $d\mathfrak{S}$ is the scalar element of volume,

$$\int\int \mathbf{f}_n dS = \int\int\int \text{div }\mathbf{f}\,d\mathfrak{S}.$$

In our notation, if $\mathrm{d}\boldsymbol{\mathfrak{S}}$ denotes vector element of volume, this becomes

$$\int\int (d\mathbf{S}\times\mathbf{f})^* = \int\int\int d\mathfrak{S}\nabla\cdot\mathbf{f} = \int\int\int d\boldsymbol{\mathfrak{S}}^*\nabla\cdot\mathbf{f},$$

which, by transformation by (24) and (32), becomes

$$\int\int d\mathbf{S}\times\mathbf{f} = \int\int\int (d\boldsymbol{\mathfrak{S}}\cdot\nabla)\times\mathbf{f}. \tag{63}$$

As an example of a similar formula involving a scalar function f, we may take the familiar theorem of hydrodynamics that the surface integral of the pressure is equal to the volume integral of the gradient of the pressure f. This is usually written as

$$\int\int f\mathbf{n}dS = \int\int\int \text{grad} f\, d\mathfrak{S},$$

but in our notation becomes

$$\int\int d\mathbf{s}f = \int\int\int d\boldsymbol{\mathfrak{S}}\cdot(\nabla f) = \int\int\int (d\boldsymbol{\mathfrak{S}}\cdot\nabla)f.$$

42. All these formulas lead us to suspect the existence of a single operational equation which is valid when applied to scalar functions and to any vector functions whether with the symbol (•) or (×). This would have the form

$$\int_{(p)} d\sigma_p() = (-1)^p \int_{(p+1)} (d\sigma_{(p+1)}\cdot\Diamond)\,(), \tag{64}$$

where $d\sigma_p$ is the p-vector element of a closed spread bounding a spread of $p+1$ dimensions. We may extend this equation to four (or more) dimensions, and demonstrate its validity as follows.

It will perhaps be sufficient to give the proof of the formula in case the $(p+1)$-spread is a rectangular parallelepiped with $p+1$ pairs of opposite faces. For let

$$d\sigma_{(p+1)} = \mathbf{k}_{123\ldots p+1}\, dx_1 dx_2 dx_3 \ldots dx_{p+1}.$$

Then, by the rules for multiplication,

$$\int_{(p+1)} d\sigma_{(p+1)}\cdot\Diamond = (-1)^p \int_{(p+1)} \Big[dx_2 dx_3 \ldots dx_{p+1}\,\mathbf{k}_{23\ldots p+1} dx_1 \frac{\partial}{\partial x_1} - dx_1 dx_3 \ldots dx_{p+1}\,\mathbf{k}_{13\ldots p+1}\, dx_2 \frac{\partial}{\partial x_2} + \ldots \Big].$$

The partial integrations may now be effected upon the right, and leave

$$\int_{(p+1)} d\sigma_{(p+1)}\cdot\Diamond = (-1)^p \int_{(p)} d\sigma_{(p)},$$

if it be remembered that $\mathbf{k}_{23\ldots p+1}$, $-\mathbf{k}_{13\ldots p+1}$, . . . are the positive faces perpendicular to $\mathbf{k}_1$, $\mathbf{k}_2$, . . .

It will be evident from this mode of proof that (64) is valid both for Euclidean and for our non-Euclidean geometry. The equation may be put in another form by the aid of rules previously given.[37]

$$\int_{(p)} d\sigma_p{}^*() = \int_{(p+1)} d\sigma_{(p+1)}{}^* \times \Diamond\,(). \tag{65}$$

In four dimensions a large number of special formulas may be obtained by applying our operational equation to scalars and to vectors of any denomination with either symbol of multiplication. As examples we may write the formulas corresponding to Stokes's and Gauss's theorems. Let $p = 1$ and apply the operator by inner multiplication to a 1-vector function. Then

$$\int d\mathbf{s}\cdot\mathbf{f} = -\int\!\!\int (d\mathbf{S}\cdot\Diamond)\cdot\mathbf{f} = \int\!\!\int d\mathbf{S}\cdot(\Diamond\times\mathbf{f}).$$

This is the extended Stokes's theorem. Again let $p = 3$ and apply the operator by outer multiplication to a 1-vector function. Then

$$\int\!\!\int\!\!\int d\mathfrak{S}\times\mathbf{f} = -\int\!\!\int\!\!\int\!\!\int (d\Sigma\cdot\Diamond)\times\mathbf{f} = -\int\!\!\int\!\!\int\!\!\int d\Sigma(\Diamond\cdot\mathbf{f}).$$

This is the extended Gauss's theorem, where $d\Sigma$ represents a differential (pseudo-scalar) element of four dimensional volume.

In these cases also the same equations apply in Euclidean and in our non-Euclidean space. If, however, we write these two equations in non-vectorial form, they become in the non-Euclidean case

$$\begin{aligned}\int (f_1dx_1 + f_2dx_2 + f_3dx_3 - f_4dx_4) \\ = \int\!\!\int \Big[\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2dx_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_3dx_1 \\ + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1dx_2 - \left(\frac{\partial f_4}{\partial x_1} + \frac{\partial f_1}{\partial x_4}\right) dx_1dx_4 \\ - \left(\frac{\partial f_4}{\partial x_2} + \frac{\partial f_2}{\partial x_4}\right) dx_2dx_4 - \left(\frac{\partial f_4}{\partial x_3} + \frac{\partial f_3}{\partial x_4}\right) dx_3dx_4 \Big]\end{aligned}$$

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[37] This equation embraces both of the operational equations given by Gibbs in §§ 164–5 of his pamphlet Vector Analysis (1884) reprinted in his Scientific Papers, **2**. In case $p + 1$ is equal to n, the number of dimensions of space, then $d\sigma_{(p+1)}{}^*$ is a scalar and the equation has no meaning unless we adopt the convention $m\times\alpha = m\alpha$, where m is a scalar and α any vector. This convention would lead to no contradiction, and might occasionally be useful.

and

$$\int\!\!\int\!\!\int\!\!\int (f_1 dx_2 dx_3 dx_4 + f_2\, dx_3 dx_1 dx_4 + f_3 dx_1 dx_2 dx_4 - f_4 dx_1 dx_2 dx_3)$$
$$= \int\!\!\int\!\!\int\!\!\int \left[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} + \frac{\partial f_4}{\partial x_4}\right] dx_1 dx_2 dx_3 dx_4.$$

The theorems may be used to demonstrate in a vectorial manner such an equation as (52), $\Diamond\cdot(\Diamond\cdot\mathbf{F}) = 0$. For

$$\int\!\!\int\!\!\int\!\!\int d\Sigma \Diamond\cdot(\Diamond\cdot\mathbf{F}) = -\int\!\!\int\!\!\int d\mathbf{S}\times(\Diamond\cdot\mathbf{F})$$
$$= \int\!\!\int\!\!\int (d\mathbf{S}\cdot\Diamond)\times\mathbf{F} = \int\!\!\int d\mathbf{s}\times\mathbf{F}.$$

As the final integral extends over the *boundary* of the *closed* three dimensional spread which bounds the given region of four dimensions, the final integral vanishes, since the closed spread has no boundary.

Geometric Vector Fields.

43. The idea of a vector field is ordinarily associated with concepts such as those of force or momentum, which are not wholly geometrical in character; but it is perfectly possible to construct vector fields which are purely geometrical. Thus in ordinary geometry we may derive a vector field, when a single point is given, by constructing at every other point the vector from that point to the given point, or that vector multiplied by any function of the distance.

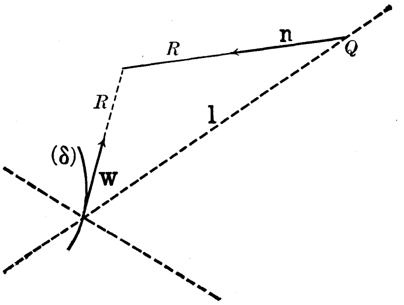

FIGURE 21.

In our non-Euclidean four dimensional space we may associate with any (δ)-curve a vector field derived from that curve in the following way. At each point of the (δ)-curve construct the forward unit tangent $\mathbf{w}$, and the forward hypercone.[38] At each point Q of these hypercones construct the vector $\mathbf{w}/R$, parallel to the vector

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[38] That half of the hypercone lying above the origin, enclosing points which will represent later times than the time of the origin, will be called the forward hypercone.

w at the vertex, and equal in magnitude to the reciprocal of the interval R along the perpendicular drawn from the point Q to that tangent produced (Figure 21). On account of analogies which will soon become apparent we shall call this vector function the extended vector potential of the given (δ)-curve.[39] We shall write

$$\mathbf{p} = \frac{\mathbf{w}}{R}. \tag{66}$$

We shall next consider the 2-vector field

$$\mathbf{P} = \Diamond\times\mathbf{p} = \left(\Diamond\frac{1}{R}\right)\times\mathbf{w} + \frac{1}{R}(\Diamond\times\mathbf{w}). \tag{67}$$

We shall consider the evaluation of $\Diamond\times\mathbf{p}$ in two steps. First we shall assume that the original (δ)-curve is a straight line. In this case **w** is constant and $\Diamond\times\mathbf{w} = 0$. If we arbitrarily take $\mathbf{k}_4$ along **w**, we may write

$$\Diamond\frac{1}{R} = \nabla\frac{1}{R} - \mathbf{k}_4\frac{\partial}{\partial x_4}\frac{1}{R} = \nabla\frac{1}{R};$$

for it is clear that a displacement parallel to **w** does not change R. It is evident that R becomes a radius vector in the 3-space perpendicular to **w**. If **n** represents a unit vector from the point Q normal to **w**, that is, in the direction in which R was measured, then by the well known formula, $\nabla R^{-1} = \mathbf{n}/R^2$. Hence

$$\Diamond\frac{1}{R} = \frac{\mathbf{n}}{R^2}.$$

And hence

$$\mathbf{P} = \Diamond\times\mathbf{p} = \frac{\mathbf{n}\times\mathbf{w}}{R^2}. \tag{68}$$

The determination of $\Diamond\cdot\mathbf{p}$ follows in precisely the same way; in each of the above formulas the symbol of inner multiplication will replace that of outer multiplication, and we find that

$$\Diamond\cdot\mathbf{p} = \frac{\mathbf{n}\cdot\mathbf{w}}{R^2} = 0, \tag{69}$$

for **n** is perpendicular to **w**.

Of all the geometrical vector fields which might have been constructed from a given (δ)-curve, we shall show later that those which we have just derived are the most fundamental (footnote § 44). The

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[39] The vector fields produced at a point by two or more (δ)-curves may be regarded as additive. The locus of all possible singular lines **l** drawn (as in Fig. 21) from (δ)-curves to a given point is the *backward* hypercone of which that point is the apex.

2-vector $\Diamond\times\mathbf{p}$ is a simple plane vector in the plane of the point Q and of $\mathbf{w}$. The 1-vector $\mathbf{p}$ has everywhere the direction of the fundamental vector $\mathbf{w}$; if $\mathbf{l}$ be the singular vector from the vertex of the cone to the point Q, the scalar product $\mathbf{l}\cdot\mathbf{p}$ is constant. In fact

$$\mathbf{p} = -\frac{\mathbf{w}}{\mathbf{l}\cdot\mathbf{w}}, \qquad \mathbf{P} = \frac{\mathbf{l}\times\mathbf{w}}{(\mathbf{l}\cdot\mathbf{w})^3} \tag{70}$$

are the expressions for the fields in terms of $\mathbf{l}$ and $\mathbf{w}$.

Let us now choose arbitrarily a time-axis $\mathbf{k}_4$, and then the perpendicular planoid is our three dimensional space. We may resolve our 1-vector and 2-vector fields as follows.

$$\begin{aligned} \mathbf{p} &= -\frac{\mathbf{w}}{\mathbf{l}\cdot\mathbf{w}} = -\frac{\mathbf{v}+\mathbf{k}_4}{(\mathbf{l}_s + l_4\mathbf{k}_4)\cdot(\mathbf{v}+\mathbf{k}_4)}, \\ \mathbf{p} &= \mathbf{p}_s + p_4\mathbf{k}_4 = \frac{\mathbf{v}}{l_4 - \mathbf{l}_s\cdot\mathbf{v}} + \frac{\mathbf{k}_4}{l_4 - \mathbf{l}_s\cdot\mathbf{v}}. \end{aligned} \tag{71}$$

where $\mathbf{l}_s$ and $\mathbf{p}_s$ are the space components of $\mathbf{l}$ and $\mathbf{p}$. As $\mathbf{l}$ is a singular vector, l_4 is equal to the magnitude of $\mathbf{l}_s$.

$$\begin{aligned} \mathbf{P} &= \frac{\mathbf{l}\times\mathbf{w}}{(\mathbf{l}\cdot\mathbf{w})^3} = -(1-v^2)\frac{(\mathbf{l}_s + l_4\mathbf{k}_4)\times(\mathbf{v}+\mathbf{k}_4)}{(l_4 - \mathbf{l}_s\cdot\mathbf{v})^3} \\ \mathbf{P} &= -\frac{(1-v^2)\,\mathbf{l}_s\times\mathbf{v}}{(l_4 - \mathbf{l}_s\cdot\mathbf{v})^3} - \frac{(1-v^2)\,(\mathbf{l}_s - l_4\mathbf{v})\times\mathbf{k}_4}{(l_4 - \mathbf{l}_s\cdot\mathbf{v})^3}. \end{aligned} \tag{72}$$

Of these two planes into which $\mathbf{P}$ is now resolved, the first lies in "space" and the second passes through the time axis and is perpendicular to "space."

We shall attempt to show with the aid of a diagram (Figure 22) the geometrical significance of the various terms which we have employed in the above formulas. The origin, that is, the vertex of the hypercone, is any chosen point O on the given (δ)-line $\mathbf{w}$. A point upon the forward hypercone is Q, and $\mathbf{l}$ is the element OQ. The unit vector $\mathbf{n}$ is drawn along QJ from Q towards and perpendicular to the vector $\mathbf{w}$. The intervals OJ and QJ are equal, and equal to $R = -\mathbf{l}\cdot\mathbf{w}$. The vector $\mathbf{p}$ drawn at Q parallel to $\mathbf{w}$ and of magnitude $1/R$ is the extended vector potential at Q due to $\mathbf{w}$. The 2-vector $\mathbf{P}$ lies in the plane OJQ, and is equal in magnitude to $1/R^2$. The arbitrarily chosen time-axis is $\mathbf{k}_4$, and on the planoid perpendicular to $\mathbf{k}_4$ (that is, on "space") the vector $\mathbf{l}$ projects into $\mathbf{l}_s = O'Q$. The intersection of the line of $\mathbf{w}$ with the planoid is G (the point of the line $\mathbf{w}$ which is simultaneous with Q). Similarly O' is the intersection of $\mathbf{k}_4$ with the planoid. The line $OO' = l_4$ represents the lapse of time between O and O'; and this is equal in magnitude to $O'Q$ or $\mathbf{l}_s$, the space component of $\mathbf{l}$. The interval $OG = l_4 \sqrt{1 - v^2}$ and the interval $O'G = l_4 v = l_s v$. The direction $\mathbf{w}$ projects into the direction $\mathbf{v}$. Hence as a vector, $O'G$ is equal to $l_s \mathbf{v}$. The quantity $\mathbf{l}_s \cdot \mathbf{v} = O'F$ may be obtained by dropping a perpendicular from G to $O'Q$. The interval FQ is then $l_s - \mathbf{l}_s \cdot \mathbf{v}$ or $l_4 - \mathbf{l}_s \cdot \mathbf{v}$, the expression which occurs in the denominators. The vector $GQ = \mathbf{r}$ is clearly $\mathbf{l}_s - l_s \mathbf{v}$ or $\mathbf{l}_s - l_4 \mathbf{v}$.

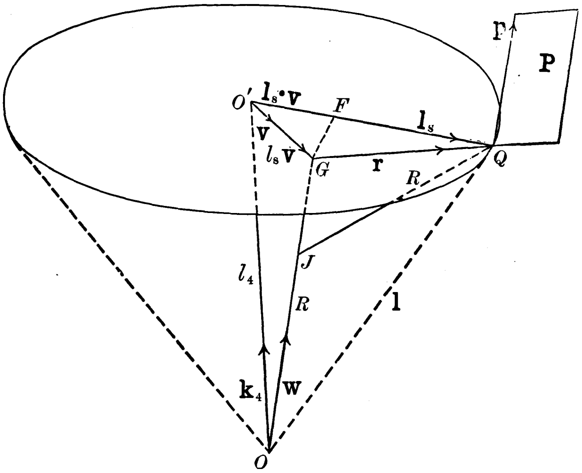

FIGURE 22.

44. We shall now remove the restriction that the (δ)-curve which gives rise to the potential $\mathbf{p} = \mathbf{w}/R = -\mathbf{w}/(\mathbf{l} \cdot \mathbf{w})$ is rectilinear, and consider the general case of any (δ)-curve. For the sake of simplicity in this complex problem we shall use dyadic notation (see appendix § 61, ff.). The results, however, might all be obtained by means of the more elementary geometric and vector methods.

We may write

$$\Diamond \mathbf{p} = \Diamond \frac{\mathbf{w}}{R} = \left(\Diamond \frac{1}{R}\right) \mathbf{w} + \frac{1}{R} \Diamond \mathbf{w} = -\frac{1}{R^2} (\Diamond R)\, \mathbf{w} + \frac{1}{R} \Diamond \mathbf{w}.$$

Now $\Diamond \mathbf{w}$ is defined so as to satisfy the relation $d\mathbf{r} \cdot \Diamond \mathbf{w} = d\mathbf{w}$. A displacement (Figure 23) $d\mathbf{r} = \mathbf{w}\, ds$ parallel to $\mathbf{w}$, makes a change $d\mathbf{w} = \mathbf{c}\, ds$. A displacement $d\mathbf{r}$ along the vector $\mathbf{l}$ (Figure 24) introduces no change in **w**, and in like manner a displacement $d\mathbf{r}$ in the plane perpendicular to that of **w** and **l** does not affect **w**. Hence we may write

$$\Diamond\,\mathbf{w} = \frac{1}{\mathbf{l}\cdot\mathbf{w}}\,\mathbf{c} = -\frac{1}{R}\,\mathbf{lc}. \tag{73}$$

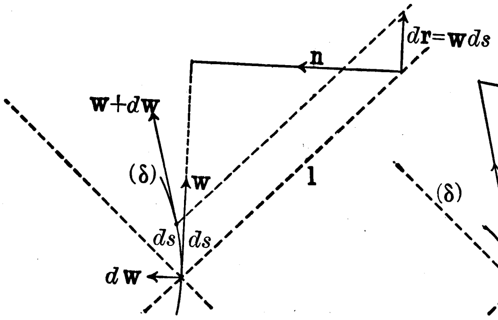

FIGURE 23.

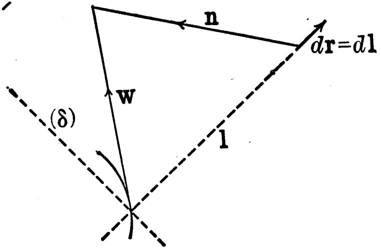

FIGURE 24.

To compute $\Diamond R = -\Diamond\,(\mathbf{l}\cdot\mathbf{w})$, we may write

$$\Diamond\,(\mathbf{l}\cdot\mathbf{w}) = (\Diamond\,\mathbf{l})\cdot\mathbf{w} + (\Diamond\,\mathbf{w})\cdot\mathbf{l}.$$

Here $\Diamond\mathbf{w}$ is already known. To find $\Diamond\mathbf{l}$ observe that $d\mathbf{l} = d\mathbf{r}\cdot\Diamond\mathbf{l}$ is equal to $d\mathbf{r}$ when $d\mathbf{r}$ is along **l** (Figure 24). Further if $d\mathbf{r}$ is elsewhere in the hypercone, for instance, in the plane perpendicular to that of **l** and **w** then also $d\mathbf{l} = d\mathbf{r}$. But when $d\mathbf{r} = \mathbf{w}ds$ is along **w** the differential $d\mathbf{l}$ vanishes. Hence we may write

$$\Diamond\,\mathbf{l} = \mathrm{I} - \frac{1}{\mathbf{l}\cdot\mathbf{w}}\,\mathbf{w} = \mathrm{I} + \frac{1}{R}\,\mathbf{lw}, \tag{74}$$

where I is the idemfactor. Thus we have

$$\Diamond\,(\mathbf{l}\cdot\mathbf{w}) = \left(\mathrm{I} + \frac{1}{R}\,\mathbf{lw}\right)\cdot\mathbf{w} - \frac{1}{R}\,\mathbf{lc}\cdot\mathbf{l},$$

or, performing the multiplication by **w**,

$$\Diamond R = -\Diamond\,(\mathbf{l}\cdot\mathbf{w}) = -\,\mathbf{w} + \frac{1 + \mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}. \tag{75}$$

From this it follows at once that

$$\begin{aligned} \Diamond\,\mathbf{p} &= -\frac{1}{R^2}(\Diamond R)\,\mathbf{w} + \frac{1}{R}\Diamond\mathbf{w} \\ &= -\frac{1}{R^2}\left(\mathbf{lc} + \frac{1 + \mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{lw} - \mathbf{ww}\right). \end{aligned} \tag{76}$$

The two expressions $\Diamond\times\mathbf{p}$ and $\Diamond\cdot\mathbf{p}$ may now be obtained by inserting the cross and dot in $\Diamond\mathbf{p}$. Hence

$$\Diamond\times\mathbf{p} = -\frac{1}{R^2}\left(\mathbf{l}\times\mathbf{c} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\times\mathbf{w}\right), \tag{77}$$

$$\Diamond\cdot\mathbf{p} = -\frac{1}{R^2}\left(\mathbf{l}\cdot\mathbf{c} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\cdot\mathbf{w} + 1\right) = 0. \tag{78}$$

Here also $\Diamond\cdot\mathbf{p}$ vanishes, since $\mathbf{l}\cdot\mathbf{w} = -R$.

As $\mathbf{l}$ varies with R, the parts of $\Diamond\times\mathbf{p}$ may be separated into one which varies as R^{-1} and one which varies as R^{-2}, namely.

$$\mathbf{P} = \Diamond\times\mathbf{p} = -\frac{1}{R^2}\left(\mathbf{l}\times\mathbf{c} + \frac{\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\times\mathbf{w}\right) - \frac{1}{R^3}\,\mathbf{l}\times\mathbf{w}. \tag{79}$$

This may be brought out most clearly by expressing $\mathbf{l}$ as

$$\mathbf{l} = R(\mathbf{w} - \mathbf{n}), \tag{80}$$

where $\mathbf{n}$ is a unit vector from Q perpendicular to $\mathbf{w}$.

$$\mathbf{P} = -\frac{1}{R}\,[\mathbf{w}\times\mathbf{c} - \mathbf{n}\times\mathbf{c} + \mathbf{n}\cdot\mathbf{c}\;\mathbf{n}\times\mathbf{w}] + \frac{1}{R^2}\,\mathbf{n}\times\mathbf{w}. \tag{81}$$

Another manner of expressing $\mathbf{P}$ is

$$\mathbf{P} = -\frac{1}{R^3}\,\mathbf{l}\times[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})] - \frac{1}{R^3}\,\mathbf{l}\times\mathbf{w} \tag{82}$$

or

$$\mathbf{P} = -\frac{1}{R^3}\,(\mathbf{l}\times\mathbf{w}\times\mathbf{c})\cdot\mathbf{l} - \frac{1}{R^3}\,\mathbf{l}\times\mathbf{w}. \tag{83}$$

Any of these forms of $\mathbf{P}$ shows, what perhaps appears clearest from (82), that the part of $\mathbf{P}$ which varies inversely as R is a singular plane, through the element $\mathbf{l}$ and cutting the plane of $\mathbf{w}\times\mathbf{c}$; for $\mathbf{l}\times[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})]$ is a plane through $\mathbf{l}$ and the vector $\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})$ (in $\mathbf{w}\times\mathbf{c}$), and the inner product of the plane by itself is readily shown to be zero.

In a similar manner we may calculate $\Diamond\mathbf{P}$, a dyadic with its first vectors 1-vectors and its second vectors 2-vectors. The differentiation requires nothing new except $\Diamond\mathbf{c}$. And by the same reasoning applied to find $\Diamond\mathbf{w}$, it appears that

$$\Diamond\,\mathbf{c} = \frac{1}{\mathbf{l}\cdot\mathbf{w}}\frac{d\mathbf{c}}{ds} = -\frac{1}{R}\frac{d\mathbf{c}}{ds}. \tag{84}$$

Hence $\Diamond\mathbf{c}$ brings in, as might be expected, the rate of change of curvature, just as $\Diamond\mathbf{w}$ brought in the curvature. We have

$$\Diamond\,\mathbf{P} = \Diamond\,(\Diamond\times\mathbf{p}) = \Diamond\left(-\frac{\mathbf{l}\times\mathbf{c}}{R^2} - \frac{1+\mathbf{l}\cdot\mathbf{c}}{R^3}\,\mathbf{l}\times\mathbf{w}\right)$$

$$= \frac{2}{R^3}\left(-\mathbf{w} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\right)\mathbf{l}\times\mathbf{c} - \frac{1}{R^2}\left(\mathrm{I} + \frac{1}{R}\,\mathbf{lw}\right)\times\mathbf{c} + \frac{1}{R^2}\left(-\frac{1}{R}\frac{d\mathbf{c}}{ds}\right)\times\mathbf{l}$$

$$+ \frac{3}{R^4}(1+\mathbf{l}\cdot\mathbf{c})\left(-\mathbf{w} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\right)\mathbf{l}\times\mathbf{w} - \frac{1}{R^3}\left(\mathbf{c} - \frac{1}{R}\frac{d\mathbf{c}}{ds}\cdot\mathbf{l}\right)\mathbf{l}\times\mathbf{w}$$

$$- \frac{1+\mathbf{l}\cdot\mathbf{c}}{R^3}\left(\mathrm{I} + \frac{1}{R}\,\mathbf{lw}\right)\times\mathbf{w} - \frac{1+\mathbf{l}\cdot\mathbf{c}}{R^3}\,\frac{1}{R}\,\mathbf{c}\times\mathbf{l}.$$

In this expression the product indicated by the cross is always performed first, regardless of the parentheses. If now the cross be inserted to find $\Diamond\times\Diamond\times\mathbf{p}$, the result $\Diamond\times\Diamond\times\mathbf{p} = 0$ is obtained, as required by equation (51). Moreover, if the dot be inserted so as to find $\Diamond\cdot(\Diamond\times\mathbf{p})$, the result is also

$$\Diamond\cdot\Diamond\times\mathbf{p} = 0. \tag{85}$$

We have, of course, proved this theorem only for points lying off the given (δ)-curve.

We have the mathematical relation (55), namely,

$$\Diamond\cdot\Diamond\times\mathbf{p} = \Diamond\,(\Diamond\cdot\mathbf{p}) - (\Diamond\cdot\Diamond)\,\mathbf{p}.$$

But we have seen that $\Diamond\cdot\mathbf{p} = 0$, and therefore

$$\Diamond\cdot\Diamond\,\mathbf{p} = \Diamond^2\mathbf{p} = 0. \tag{86}$$

The existence of this extended Laplacian equation justifies the use of the term potential[40] for $\mathbf{p}$.

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[40] It is interesting to enquire what form the potential $\mathbf{p}$ might be given other than $\mathbf{w}/R$. Suppose that $\mathbf{p}$ should be independent of the curvature of the (δ)-curve. The only vectors then entering into the determination of $\mathbf{p}$ at any point Q would be $\mathbf{w}$ and $\mathbf{l}$. The only possible form of a 1-vector potential would therefore be

$$\mathbf{p} = \varphi\,(R)\,\mathbf{w} + f\,(R)\,\mathbf{l},$$

where $R = -\mathbf{l}\cdot\mathbf{w}$. The expression for $\Diamond\mathbf{p}$ becomes

$$\Diamond\mathbf{p} = \varphi'\,(R)\left(-\mathbf{w} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\right)\mathbf{w} - \varphi\,(R)\,\frac{1}{R}\,\mathbf{lc}$$

$$+ f'\,(R)\left(-\mathbf{w} + \frac{1+\mathbf{l}\cdot\mathbf{c}}{R}\,\mathbf{l}\right)\mathbf{l} + f\,(R)\left(\mathrm{I} + \frac{1}{R}\,\mathbf{lw}\right)\cdot$$

ELECTROMAGNETICS AND MECHANICS.

The Continuous and Discontinuous in Physics.

45. It has been customary in physics to regard a fluid as composed of discrete particles (as in the kinetic theory) or as a continuum (as in hydrodynamics) according to the nature of the problem under investigation; it has been assumed that even if a fluid were made up of discrete particles, it could be treated as a continuum for the sake of convenience in applying the laws of mathematical analysis. For example we introduce the concept of density which may have no real exact physical significance, but which by the method of averages yields apparently correct results. Provided that the particles in a discontinuous assemblage are sufficiently small, numerous, and regularly distributed, it is assumed that any assemblage of discrete particles can be replaced without loss of mathematical rigor by a continuum.

However, when we investigate problems of this character in the light of our four dimensional geometry, we are led to the striking conclusion that in some cases it is impossible, except by methods which are unwarrantably arbitrary, to replace a discontinuous by a continuous distribution and vice versa. Especially we shall see that this is the case with radiant energy, a conclusion which is particularly

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Hence

$$\Diamond \cdot \mathbf{p} = -\mathbf{l} \cdot \mathbf{c} \left(\varphi'(R) + \frac{1}{R} \varphi(R) \right) + \left(Rf'(R) + 3f(R) \right).$$

If $\Diamond \cdot \mathbf{p}$ is to vanish regardless of the curvature of the (δ)-curve, then

$$\varphi'(R) + \frac{1}{R} \varphi(R) = 0, \qquad Rf'(R) + 3f(R) = 0.$$

The integration of these equations determines φ and f as

$$\varphi = \frac{A}{R}, \qquad f = \frac{B}{R^3},$$

where A and B are constants. The expression for $\Diamond \times \mathbf{p}$ is

$$\Diamond \times \mathbf{p} = -\frac{A}{R^2} \left(\mathbf{l} \times \mathbf{c} + \frac{1 + \mathbf{l} \cdot \mathbf{c}}{R} \mathbf{l} \times \mathbf{w} \right) - \frac{2B}{R^4} \mathbf{l} \times \mathbf{w}.$$

The calculation of $\Diamond \cdot \Diamond \times \mathbf{p} = -\Diamond \cdot \Diamond \mathbf{p}$ gives

$$\Diamond \cdot \Diamond \mathbf{p} = 2B \left(\frac{\mathbf{w}}{R^4} + 3 \frac{\mathbf{l} \cdot \mathbf{c}}{R^5} \mathbf{l} \right).$$

It therefore appears impossible to satisfy $\Diamond \cdot \mathbf{p} = 0$ and $\Diamond \cdot \Diamond \mathbf{p} = 0$ with any other form of potential, dependent only on **l** and **w**, than the one chosen.

notable when taken in connection with the recent theories regarding the constitution of light, embodied in the quantum hypothesis.

Let us for simplicity first consider such cases as arise in our two dimensional geometry. Consider a material rod of infinitesimal cross section moving uniformly in its own direction. Suppose now that we regard this rod as made up of discrete particles. Then in our geometrical representation each particle will give rise to a vector of extended momentum $m_0\mathbf{w}$, and these vectors will all be parallel. The whole space-time locus of the rod will be a set of parallel (δ)-lines. The rod as a spacial object possessing length has no meaning until a definite set of space-time axes have been chosen, and this choice is arbitrary. There is, however, one such choice which is unique, and that is the selection of the time-axis along $\mathbf{w}$, and the space-axis perpendicular thereto. In this system the mass of each particle is its m_0, and the sum of the m_0's of any segment of the rod divided by the length of the segment is the average density. If the particles are sufficiently numerous, we may regard the rod as continuous and replace conceptually the locus of the rod as a set of discrete (δ)-lines by a vector field continuous between the two (δ)-lines which mark the termini of the rod, and represented at each point by a vector parallel to $\mathbf{w}$ and equal in magnitude to the density at that point. This is the density as it appears to an observer at rest with respect to the rod, and may be called μ_0. The vector $\mu_0\mathbf{w}$ has therefore a definite four dimensional significance. Its projections on any arbitrarily chosen space and time axes are, however, not respectively the density of momentum and mass in that system. For

$$\mu_0\mathbf{w} = \frac{\mu_0}{\sqrt{1-v^2}}(\mathbf{v} + \mathbf{k}_4). \tag{87}$$

But μ, the density in this system, is not equal to $\mu_0/\sqrt{1-v^2}$, but

$$\mu = \frac{\mu_0}{1-v^2} \tag{88}$$

as the units of mass and length both change with a change of axes.

Conversely we may replace a continuous by a discrete distribution. Let us consider a continuous vector field $\mathbf{f}$ of (δ)-lines. Then any region of this field, embraced between two (δ)-lines sufficiently near together, may be replaced by one or several parallel (δ)-vectors, of which the sum is equal to $\mathbf{f}$ multiplied by the length of the line drawn between and perpendicular to the boundary (δ)-lines. We may also use another construction which is essentially identical with this. Let $d\mathbf{r}$ be any vector drawn from one boundary line to the other. Then $(d\mathbf{r}\times\mathbf{f})^*\mathbf{f}/f$ is the same vector as the one just obtained. Although the method of obtaining this vector may seem somewhat artificial, the vector is, however, a definite vector obtainable from the field without any choice of axes.

46. These methods fail completely when the vector field is composed of singular vectors. Let us consider instead of a material rod, a segment of a uniform ray of light. If this can be represented by a continuous vector field bounded by two lines representing the loci of the termini of the segment then all these vectors must be singular. Let $\mathbf{l}$ be (Figure 25) the value of the vector throughout the field. It is evident that we cannot, as in the former case, draw any line across the field perpendicular to $\mathbf{l}$. The second method likewise fails because it would involve the magnitude of $\mathbf{l}$ which is zero. Moreover it can be stated that there is no method whatever, independent of any choice of axes, which will enable us to change from this continuous distribution of the light to a set of light particles. Conversely it is equally true that given a system of light particles moving in a single ray it is quite impossible to replace them by means of any continuous distribution, and this is true no matter how small and numerous and close together these particles are. This statement regarding singular vectors will be seen to hold also in space of higher dimensions,[41] and is of fundamental importance.

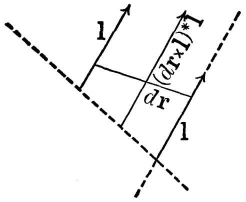

FIGURE 25.

While it is impossible, therefore, to find continuous and discontinuous distributions of singular vectors which are equivalent to one another, it is possible to obtain by four dimensional methods out of a specified region of a singular vector field a single vector or group of discrete vectors uniquely determined by that vector field but quadratic instead of linear in the vectors of the field. Consider any portion of the field bounded by two singular vectors sufficiently near together. Let $\mathbf{l}$ be the vector of the field, and then if $d\mathbf{r}$ is any vector drawn from one boundary to the other (Figure 25), the 2-vector $d\mathbf{r}\times\mathbf{l}$ is independent of the way in which $d\mathbf{r}$ was drawn and the 1-vector $(d\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ is determined, and is in a certain sense representative of the region of the field chosen.

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[41] In the case of the peculiar geometry of a singular plane (§ 31), the interval dr from one singular line to another is independent of the direction of $d\mathbf{r}$. It is therefore possible to replace the field $\mathbf{l}$ between two boundary lines by the single vector $\mathbf{l}dr$ linear in $\mathbf{l}$. Thus there are exceptional singular fields in higher dimensions for which the passage from continuous to discrete and vice versa may be accomplished.

It may be of interest to obtain the projection of $\mathbf{l}$ and $(d\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ upon two sets of axes $\mathbf{k}_1$, $\mathbf{k}_4$ and $\mathbf{k}_1'$, $\mathbf{k}_4'$ where the angle from $\mathbf{k}_4$ to $\mathbf{k}_4'$ is $\phi = \tanh^{-1} v$. Let the vector $\mathbf{l}$ be written as

$$\mathbf{l} = a\,(\mathbf{k}_1 + \mathbf{k}_4) = a'\,(\mathbf{k}_1' + \mathbf{k}_4').$$

Now by the transformation equations (7) we have

$$a' = a\,(\cosh\psi - \sinh\psi) = a\frac{1-v}{\sqrt{1-v^2}} = a\sqrt{\frac{1-v}{1+v}}.$$

Hence the ratio of the components of $\mathbf{l}$ along the new axes to the components along the old axes is $\sqrt{1-v}/\sqrt{1+v}$. But $(d\mathbf{r}\times\mathbf{l})^*$ is a member independent of any system of axis. Hence the ratio for $(d\mathbf{r}\times\mathbf{l})^*\,\mathbf{l}$ is the same as that for $\mathbf{l}$.

Now while it is impossible by any four dimensional methods to redistribute the vector $(\mathrm{d}\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ as a continuous vector field, it is always possible after arbitrary axes of space and time have been chosen to make such a distribution. Thus if between the two boundary lines $d\mathbf{r}$ be taken parallel to $\mathbf{k}_1$ and $d\mathbf{r}'$ parallel to $\mathbf{k}'_1$, then as before $d\mathbf{r}\times\mathbf{l} = d\mathbf{r}'\times\mathbf{l}$. By taking the complement of both sides and applying (24), then, since $\mathbf{l}$ is its own complement, we find $d\mathbf{r}\cdot\mathbf{l} = d\mathbf{r}'\cdot\mathbf{l}$. But $d\mathbf{r}\cdot\mathbf{l}$ is equal to $a d\mathbf{r}\cdot\mathbf{k}_1 = a dr$, and $d\mathbf{r}'\cdot\mathbf{l} = a'dr'$. Hence $dr/dr' = a'/a$. Thus the *density* of the components of the vector $(d\mathbf{r}\times\mathbf{l})^*\mathbf{l}$ in the one case is to the density of the components in the other case as a^2 is to a'^2, equal to $(1 - v)/(1 + v)$. Thus while we have seen that the energy and momentum of a light-particle (§ 24) appear different in the ratio $\sqrt{1-v}/\sqrt{1+v}$ to two observers, if the energy and momentum are regarded as distributed their densities will appear different to the two observers in the ratio $(1-v)/(1 + v)$.

Let us proceed at once to the discussion of similar problems arising in space of four dimensions. Here also it is possible to pass at will from a consideration of continuous 1-vector fields to a consideration of equivalent discontinuous distributions of 1-vectors in the case of all non-singular vectors, by an extension of either of the methods which we have used in two dimensional space. Thus if a region of the field is cut out by a (hyper-) tube of lines parallel to the vector of the field, then the original vector multiplied by the volume of intersection of a perpendicular planoid is a single vector (or the sum of a group of vectors) which may replace the original field within the tube. Or if **f** represents the vector field and $d\mathfrak{S}$ the 3-vector cut off on any planoid by the tube, then the same result as before may be obtained by the operation $(d\mathfrak{S}\times\mathbf{f})^*\mathbf{f}/f$.

In the case of singular vectors we encounter the same difficulties as in two dimensions. Let us consider a field of singular 1-vectors **l**, and a portion of this field cut off by a small tube of lines parallel to **l**. A little consideration shows that it is impossible by any means whatever to replace this portion of the field by a single equivalent vector along **l**. It is possible, however, as before to obtain a single vector quadratic in **l** and determined by the given portion of the field. Let $d\mathfrak{S}$ be the 3-vector volume cut off on any planoid by the tube. Then $(d\mathfrak{S}\times\mathbf{l})$ is independent of the planoid chosen, and $(d\mathfrak{S}\times\mathbf{l})^*\,\mathbf{l} = d\mathbf{g}$ is the vector thus determined.

47. Now it is impossible to distribute the vector just obtained over that portion of the four dimensional spread which has given rise to it. But there is, nevertheless, in one case another kind of distribution which is possible and which possesses considerable interest. In order to introduce the somewhat difficult construction which is necessary in this case let us investigate first a particular type of singular vector field in three dimensions. Let $d\mathbf{s}$ be a small vector segment of a (δ)-curve. Each point of this segment determines a forward cone. The field which we wish to consider is such that at each point the vector **l** is along an element of the cone and of any interval which is a continuous function of position. This construction gives a limited field bounded by the two forward cones from the termini of the segment $d\mathbf{s}$. Let a plane cut across the two cones. The region of this plane intercepted between the two boundary cones is the surface lying between two nearly concentric circles. Let $d\mathbf{S}$ be an element of this surface. Now just as before the vector $(d\mathbf{S}\times\mathbf{l})^*\mathbf{l} = d\mathbf{g}$ may be formed and is different for each element $d\mathbf{S}$. The singular lines drawn from all the points bounding $d\mathbf{S}$ to the corresponding points of the segment $d\mathbf{s}$ determine a sort of tube of nearly parallel singular lines. The value of $d\mathbf{g}$ for each tube is at each point independent of the particular position of the plane through that point whose intersection with the tube is $d\mathbf{S}$. If therefore the whole field is divided up into an infinite number of such tubes, the infinitesimal vectors of the second order in **l** obtained for the several tubes are at each point independent of the plane which was used in constructing them.

Now it is impossible to redistribute the discrete vectors $d\mathbf{g}$ over the three dimensional field from which they were derived, but it is possible to replace them by a continuous distribution over a two dimensional spread in one of the cones. Let us assume that the infinitesimal tubes are so chosen that the elements of surface $d\mathbf{S} = d\mathbf{q}{\times}d\mathbf{r}$ are four-sided figures approximately rectangular and that the outer cone is divided into small regions lying between the elements of the cone, a, a', a'', . . . (Figure 26). In each of these small two dimensional regions we may place the corresponding vector $d\mathbf{g}$. Now any two neighboring lines drawn from a to a' are of equal interval because they lie in a singular plane between two singular lines (see preceding footnote and § 31). The vector $d\mathbf{g}/dr$ is therefore determined at each point of the cone independent of the direction of $d\mathbf{r}$. It is a vector representing a kind of density and when all the vectors $d\mathbf{g}$ are similarly treated, it is continuously distributed over the whole cone.

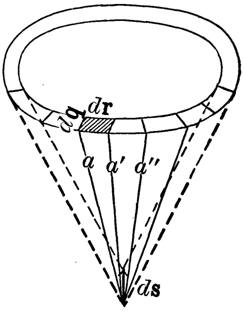

FIGURE 26.

The vector $d\mathbf{g}/dr$ is a function of the interval ds. Let us determine this relation analytically. Since $d\mathbf{S} = d\mathbf{q}{\times}d\mathbf{r}$ we may write

$$d\mathbf{g} = (d\mathbf{q}{\times}d\mathbf{r}{\times}\mathbf{l})^{*}\mathbf{l} = [(d\mathbf{q}{\times}d\mathbf{r})^{*}\cdot\mathbf{l}]\,\mathbf{l} = l_4\mathbf{l}\; dq\,dr,$$

where l_4 is the component of $\mathbf{l}$ perpendicular to $d\mathbf{q}{\times}d\mathbf{r}$; for since $d\mathbf{q}$ is perpendicular to $d\mathbf{r}$, $(d\mathbf{q}{\times}d\mathbf{r})^{*}$ is a 1-vector perpendicular to $d\mathbf{q}{\times}d\mathbf{r}$ and of magnitude $dqdr$. We therefore find $d\mathbf{g}/dr = l_4\mathbf{l}dq$. It remains to determine dq in terms of ds.

The plane of intersection having been chosen, the two circles are in general eccentric and the distance de between their centers is the projection of the segment ds upon their plane (Figure 27). If the normal to this plane makes an angle with ds whose hyperbolic tangent is v, then $de = vds/\sqrt{1 - v^2}$. The two segments cut off by the two circles on de produced are found as follows. Pass a plane through de and ds. Then AB is readily shown to be

$$ds\sqrt{1 - v}/\sqrt{1 + v}, \qquad \text{and} \qquad CD = ds\sqrt{1 + v}/\sqrt{1 - v}.$$

Then the value of dq is readily proved by Euclidean methods to be

$(1-v\cos\phi)\,ds/\sqrt{1-v^2}$, where ϕ is the angle between dq and AD. Hence

$$\frac{d\mathbf{g}}{dr}=l_4\frac{1-v\cos\phi}{\sqrt{1-v^2}}\mathbf{l}\,ds. \tag{89}$$

We have gone through this somewhat complicated calculation for the three dimensional case because of the greater ease of visualisation and because the results obtained are applicable without essential change to four dimensions. Again let $d\mathbf{s}$ be a segment of any (δ)-curve each point of which determines a forward hypercone. Let us consider the four dimensional vector field $\mathbf{l}$ bounded by the two limiting forward hypercones, $\mathbf{l}$ at every point lying along an element of one of the hypercones whose apex is on $d\mathbf{s}$. Any (γ)-planoid will intersect the limited vector field in a three dimensional volume bounded by the intersections of the two limiting hypercones with the planoid; these surfaces of intersection appear in the planoid as two nearly concentric spherical surfaces.

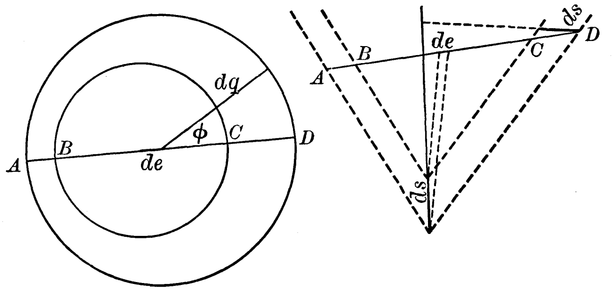

FIGURE 27.

If as before the vector field is divided into infinitesimal portions, so that the volume of intersection is divided into the infinitesimal volumes $d\mathfrak{S}$, each of which is approximately a rectangular parallelepiped, and one of the surfaces of intersection is thus divided into the infinitesimal portions dS such that $d\mathbf{q}\times d\mathbf{S}=d\mathfrak{S}$, then for each infinitesimal portion of the field we may at any point obtain as above the vector $d\mathbf{g}=(d\mathfrak{S}\times\mathbf{l})^*\mathbf{l}$. Then precisely as in the previous case [42]

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[42] In the peculiar three dimensional geometry of a tangent (singular) planoid there is one set of parallel singular lines, and every plane in the planoid is perpendicular to these lines. Every cross-section of a given tube of singular lines has the same area.

$$d\mathbf{g} = (d\mathbf{q}\times d\mathbf{S}\times\mathbf{l})^{*}\mathbf{l} = l_4\mathbf{l}\,dq\,dS, \qquad \text{and} \quad d\mathbf{g}/dS = l_4\mathbf{l}\,dq.$$

This vector is distributed uniformly over one of the hypercones and is independent of the particular planoid used in obtaining it. Then also just as before

$$\frac{d\mathbf{g}}{dS} = l_4\frac{1 - v\cos\phi}{\sqrt{1 - v^2}}\mathbf{l}\,ds, \tag{90}$$

where ϕ is the angle between **v**, which passes through the centers of the two spheres, and the line, from either center, to the chosen point upon the surface.

The Field of a Point Charge.

48. Much of recent progress in the science of electricity has been due to the introduction of the electron theory, in which electricity is regarded not as a continuum but as an assemblage of discrete particles. In Lorentz's development of this theory he has deemed it necessary, however, to regard the electron itself as distributed over a minute region of space known as the volume of the electron. This deprives the theory of some of that simplicity which it would possess if the charge of an electron could be regarded as in fact concentrated at a single point. Whether the theory of the point charge can be brought into accord with observed facts and with the laws of energy cannot at present be decided. It seems, however, highly desirable to develop this theory as far as possible. In our application of our four dimensional geometry to electricity we shall therefore consider first an electric charge as a collection of discrete charges or electrons, each of which is concentrated at a single point.

The locus of a point electron in time and space must be a (δ)-curve. If **w** is a unit tangent to such a curve, then we may consider at every point the vector $\epsilon\mathbf{w}$, where ϵ is the magnitude of the charge, negative for a negative electron, and positive for a positive electron (if such there be). It is explicitly assumed that ϵ is a constant. We shall show that the geometric fields obtained from this vector by the methods of § 43 give precisely the equations which are of importance in electromagnetic theory.

The vector **w** determines at every point of our time-space manifold the vector $\mathbf{p} = \mathbf{w}/R$. Similarly the vector $\epsilon\mathbf{w}$ determines the vector field

$$\mathbf{m} = \epsilon\mathbf{p} = \frac{\epsilon\mathbf{w}}{R} = \frac{\epsilon\mathbf{v}}{l_4 - \mathbf{l}_s\cdot\mathbf{v}} + \frac{\epsilon\mathbf{k}_4}{l_4 - \mathbf{l}_s\cdot\mathbf{v}}. \tag{91}$$

The last equality is obtained when any $\mathbf{k}_4$ axis has been arbitrarily chosen. Then $\mathbf{v}$ is the velocity of the electron and $l_4 - \mathbf{l}_s \cdot \mathbf{v}$ is the distance *FQ* in Figure 22, that is, the projection of the distance from the point of observation to the contemporaneous position of the electron (if assumed to be moving uniformly) upon the line $\mathbf{l}_s$ joining the "retarded" position of the electron to the point of observation.

We may call **m** the extended electromagnetic vector potential. Its projections on space and on the time-axis are respectively the vector potential **a** and the scalar potential ϕ,

$$\mathbf{a} = \frac{\epsilon \mathbf{v}}{l_4 - \mathbf{l}_s \cdot \mathbf{v}}, \qquad \phi = \frac{\epsilon}{l_4 - \mathbf{l}_s \cdot \mathbf{v}}, \tag{92}$$

precisely in the form first obtained by Liénard.[43] From (69) we have

$$\Diamond \cdot \mathbf{m} = \left(\nabla - \mathbf{k}_4 \frac{\partial}{\partial t}\right) \cdot (\mathbf{a} + \phi \mathbf{k}_4) = 0.$$

Hence

$$\nabla \cdot \mathbf{a} + \frac{\partial \phi}{\partial t} = 0.$$

We see therefore that the Liénard potentials are connected by the same familiar equation as connects the ordinary vector and scalar potentials. Assuming that vector fields produced by two or more electrons are additive, these equations are true for the general case.

The 2-vector field produced by an electron, whether in uniform or accelerated motion, is obtained immediately from (81)–(83).

$$\mathbf{M} = \Diamond \times \mathbf{m} = \epsilon \Diamond \times \mathbf{p} = -\frac{\epsilon}{R}[\mathbf{w} \times \mathbf{c} - \mathbf{n} \times \mathbf{c} + \mathbf{n} \cdot \mathbf{c}\ \mathbf{n} \times \mathbf{w}] + \frac{\epsilon}{R^2} \mathbf{n} \times \mathbf{w}. \tag{93}$$

Or

$$\mathbf{M} = -\frac{\epsilon}{R^3} \mathbf{l} \times [\mathbf{l} \cdot (\mathbf{w} \times \mathbf{c})] - \frac{\epsilon}{R^3} \mathbf{l} \times \mathbf{w} = -\frac{\epsilon}{R^3} (\mathbf{l} \times \mathbf{w} \times \mathbf{c}) \cdot \mathbf{l} - \frac{\epsilon}{R^3} \mathbf{l} \times \mathbf{w}. \tag{94}$$

The first term in this expression vanishes when the curvature is zero. The fact that this term is a *singular* vector has already been pointed out, and the great importance of this fact in electromagnetic theory will be pointed out later. In the second term $\mathbf{n} \times \mathbf{w}$ is the unit 2-vector determined by the line **w** and the point *Q* where the field is being discussed.

49. In case the electron is unaccelerated the equation assumes the simple form

$$\mathbf{M} = \frac{\epsilon}{R^2} \mathbf{n} \times \mathbf{w}. \tag{95}$$

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[43] Eclairage électrique, **16**, 5 (1898).

This may be expanded according to (72) when an axis of time has been chosen. Then, noting that $\mathbf{l}_s \times \mathbf{v} = (\mathbf{l}_s - l_4\mathbf{v}) \times \mathbf{v}$,

$$\mathbf{M} = -\epsilon \frac{1 - v^2}{r'^3} \mathbf{r} \times \mathbf{v} - \epsilon \frac{1 - v^2}{r'^3} \mathbf{r} \times \mathbf{k}_4. \tag{96}$$

Where $\mathbf{r}$ is the vector $\mathbf{r} = \mathbf{l}_s - l_4\mathbf{v}$ from the contemporaneous position of the charge to the point Q in the field, and $r' = l_4 - \mathbf{l}_s \cdot \mathbf{v}$. The 2-vector $\mathbf{M}$ is thus split automatically into two 2-vectors, of which one passes through the time-axis $\mathbf{k}_4$, and the other lies in the planoid $\mathbf{k}_{123}$ which constitutes ordinary space. These will be designated respectively by the letters $\mathbf{E}$ and $\mathbf{H}$. Thus

$$\mathbf{M} = \mathbf{H} + \mathbf{E}. \tag{97}$$

This separation may in all cases be made whether the field is caused by one or more electrons in constant or accelerated motion. We shall thus see that the 2-vector $\mathbf{M}$ is precisely the "Vektor zweiter Art" which Minkowski introduced to express the electric and magnetic forces.

Out of $\mathbf{H}$ and $\mathbf{E}$ spacial 1-vectors $\mathbf{h}$ and $\mathbf{e}$ may be obtained by the equations

$$\mathbf{h} = \mathbf{H} \cdot \mathbf{k}_{123}, \qquad \mathbf{e} = \mathbf{E} \cdot \mathbf{k}_4. \tag{98}$$

Then $\mathbf{h}$ is the three-dimensional complement of $\mathbf{H}$, and $\mathbf{e}$ the intersection of $\mathbf{E}$ with three-dimensional space. Evidently

$$\begin{aligned} h_1 &= H_{23}, & h_2 &= H_{31}, & h_3 &= H_{12}, \\ e_1 &= -E_{14}, & e_2 &= -E_{24}, & e_3 &= -E_{34}. \end{aligned} \tag{99}$$

Referring now to (96) we see that in the case of a uniformly moving electron

$$\mathbf{e} = \epsilon \frac{1 - v^2}{r'^3} \mathbf{r}, \qquad \mathbf{h} = -\epsilon \frac{1 - v^2}{r'^3} (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{k}_{123}, \tag{100}$$

or

$$\mathbf{v} \times \mathbf{e} = \mathbf{h} \cdot \mathbf{k}_{123} = \mathbf{H}.$$

Noting that $(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{k}_{123}$ is that which in ordinary vector analysis is known as the vector product of $\mathbf{r}$ and $\mathbf{v}$, we see that these equations are precisely the equations for the electric and magnetic forces.[44]

It may seem surprising to one who is not fully convinced of the very fundamental relationship between the four dimensional geometry of relativity and the science of mechanics that we should thus be led from simple geometrical premises to conclusions of so purely physical a character. Of course it is to be noted that while our values of $\mathbf{e}$ and $\mathbf{h}$ are identical in mathematical form with equations for electric and magnetic force, we should need some additional assumptions before actually identifying these quantities.

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[44] See Abraham, Theorie der Elektrizität, **2**, p. 88.

50. Our next step will be to show that the values of $\mathbf{e}$ and $\mathbf{h}$ derived from the 2-vector $\Diamond \times \mathbf{m} = \mathbf{M}$ are identical with the expressions for electric and magnetic force in the general case in which the electron is no longer restricted to uniform motion. We have from (94)

$$\mathbf{M} = \epsilon \mathbf{P} = -\frac{\epsilon}{R^3}(\mathbf{l} \times \mathbf{w} \times \mathbf{c}) \cdot \mathbf{l} - \frac{\epsilon}{R^3} \mathbf{l} \times \mathbf{w}. \tag{101}$$

Thus, assuming some time-axis, we see from (43) that

$$\mathbf{w} \times \mathbf{c} = \mathbf{w} \times \dot{\mathbf{v}} / (1 - v^2).$$

Then

$$\mathbf{M} = -\frac{\epsilon}{R^3} \frac{(\mathbf{l} \times \mathbf{w} \times \dot{\mathbf{v}}) \cdot \mathbf{l}}{1 - v^2} - \frac{\epsilon}{R^3} \mathbf{l} \times \mathbf{w}. \tag{102}$$

Hence

$$\mathbf{M} = -\frac{\epsilon}{R^3} \frac{\mathbf{l} \cdot \dot{\mathbf{v}}\, \mathbf{l} \times \mathbf{w} + R\, \mathbf{l} \times \dot{\mathbf{v}}}{1 - v^2} - \frac{\epsilon}{R^3} \mathbf{l} \times \mathbf{w}, \tag{103}$$

$$\begin{aligned} \mathbf{M} = &-\frac{\epsilon}{R^3} \left[\frac{\mathbf{l}_s \cdot \dot{\mathbf{v}}\, \mathbf{l}_s \times \mathbf{v}}{(1 - v^2)^{\frac{3}{2}}} + \frac{R\, \mathbf{l}_s \times \dot{\mathbf{v}}}{1 - v^2} \right] - \frac{\epsilon}{R^3} \frac{\mathbf{l}_s \times \mathbf{v}}{(1 - v^2)^{\frac{1}{2}}} \\ &-\frac{\epsilon}{R^3} \left[\frac{\mathbf{l}_s \cdot \dot{\mathbf{v}}\, (\mathbf{l}_s - l_4 \mathbf{v}) \times \mathbf{k}_4}{(1 - v^2)^{\frac{3}{2}}} - \frac{R l_4 \dot{\mathbf{v}} \times \mathbf{k}_4}{1 - v^2} \right] - \frac{\epsilon}{R^3} \frac{(\mathbf{l}_s - l_4 \mathbf{v}) \times \mathbf{k}_4}{(1 - v^2)^{\frac{1}{2}}}. \end{aligned} \tag{104}$$

Hence, if again we use $\mathbf{r} = \mathbf{l}_s - l_4 \mathbf{v}$ and $r' = l_4 - \mathbf{l}_s \cdot \mathbf{v} = R(1 - v^2)^{\frac{1}{2}}$, we have

$$\begin{aligned} \mathbf{e} &= \mathbf{E} \cdot \mathbf{k}_4 = \epsilon \left[\frac{\mathbf{l}_s \cdot \dot{\mathbf{v}}}{r'^3} \mathbf{r} - \frac{l_4 \dot{\mathbf{v}}}{r'^2} + \frac{1 - v^2}{r'^3} \mathbf{r} \right] \\ \mathbf{h} &= \mathbf{H} \cdot \mathbf{k}_{123} = -\epsilon \left[\frac{\mathbf{l}_s \cdot \dot{\mathbf{v}}\ \mathbf{r} \times \mathbf{v}}{r'^3} + \frac{\mathbf{l}_s \times \dot{\mathbf{v}}}{r'^2} + \frac{(1 - v^2) \mathbf{r} \times \mathbf{v}}{r'^3} \right] \cdot \mathbf{k}_{123}. \end{aligned} \tag{105}$$

If we look at the form in $\mathbf{l}_s - l_4 \mathbf{v}$ (104) we observe that

$$\mathbf{H} = \frac{1}{l_4} \mathbf{l}_s \times \mathbf{e}, \qquad \mathbf{E} = -\mathbf{e} \times \mathbf{k}_4. \tag{106}$$

Hence

$$\mathbf{M} = \left(\frac{1}{l_4} \mathbf{l}_s + \mathbf{k}_4 \right) \times \mathbf{e} = \frac{1}{l_4} \mathbf{l} \times \mathbf{e}. \tag{107}$$

These are the equations for the field of an accelerated electron which were obtained by Abraham and Schwarzschild.[45] It will be convenient to divide the field $\mathbf{M}$ into that part $\mathbf{M}'$ which is due to acceleration alone and that $\mathbf{M}''$ which is independent of acceleration. The former, which is the first term in any of the above expressions for $\mathbf{M}$, (101)–(103), is a singular vector field, and is the only one which is important at great distances from the electron, for it varies as $1/R$ (since $\mathbf{l}$ varies with R) whereas $\mathbf{M}''$ varies as $1/R^2$. If we divide the field $\mathbf{M}'$ into its two parts $\mathbf{M}' = \mathbf{E}' + \mathbf{H}'$, we see here also that

$$\mathbf{H}' = \frac{1}{l_4}\mathbf{l}_s\times\mathbf{e}', \qquad \mathbf{E}' = -\,\mathbf{e}'\times\mathbf{k}_4; \tag{108}$$

and since, in this case, $\mathbf{l}_s\cdot\mathbf{e}' = 0$ (as may be seen by performing the multiplication) and $\mathbf{l}_s$ is perpendicular to $\mathbf{e}'$, we find that $\mathbf{E}'$, $\mathbf{H}'$ are equal in magnitude. Moreover $\mathbf{e}'$, $\mathbf{h}'$ are equal in magnitude and perpendicular to each other and to $\mathbf{l}_s$. In other words in a radiation field the electric and magnetic forces are equal in magnitude, perpendicular to each other, and perpendicular to the "direction of propagation." All these results are geometric consequences of the fact that the 2-vector $\mathbf{M}'$ is singular.

51. In four dimensional space every singular 2-vector determines a singular 1-vector, namely, a vector pointing outward along the element of tangency of the 2-vector with a forward hypercone. This 1-vector is the complement of the 2-vector in the tangent planoid. If $\mathbf{l}'$ is the 1-vector thus determined by the 2-vector $\mathbf{M}'$, then we may write

$$\mathbf{M}' = \mathbf{u}\times\mathbf{l}',$$

where $\mathbf{u}$ is any unit vector in the plane of $\mathbf{M}'$, provided the sign of $\mathbf{u}$ be properly chosen.[46] In the case of the singular vector $\mathbf{M}'$ which we have obtained in the previous section we may write, from (94),

$$\mathbf{M}' = -\frac{\epsilon}{R^3}\,\mathbf{l}\times[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})] = -\frac{\epsilon}{R^3}\,a\mathbf{l}\times\frac{\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})}{a}, \tag{109}$$

where a is the magnitude of $\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})$ and therefore the last vector is a unit vector. Hence we may write at once for the 1-vector determined by $\mathbf{M}'$,

$$\mathbf{l}' = \frac{\epsilon}{R^3}\,a\mathbf{l}. \tag{110}$$

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[45] See Abraham, Theorie der Elektrizität, **2**, p. 95.

[46] Owing to the nature of the geometry in a singular plane, the unit vector $\mathbf{u}$ drawn from a given point always terminates on a definite singular line and thus determines the same 2-vector $\mathbf{u}\times\mathbf{l}'$ for all values of $\mathbf{u}$. (§ 31)

The value of α is, from (80),

$$\alpha = \sqrt{[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})]\cdot[\mathbf{l}\cdot(\mathbf{w}\times\mathbf{c})]} = R\sqrt{(\mathbf{n}\times\mathbf{c})\cdot(\mathbf{n}\times\mathbf{c})}. \tag{111}$$

Now the vector $\mathbf{l}'$, being a singular vector continuously distributed, can be treated by the method of § 47 to give at any point a discrete vector of the second order in $\mathbf{l}'$, namely,[47]

$$d\mathbf{g} = (d\mathfrak{S}\times\mathbf{l}')^*\mathbf{l}' \tag{112}$$

where $d\mathfrak{S}$ is the vector volume cut off on any planoid by an infinitesimal tube of singular lines parallel to $\mathbf{l}'$. If $d\mathbf{s}$ is an infinitesimal portion of the locus of the electron which gives rise to the fields $\mathbf{M}'$ and $\mathbf{l}'$, and if we consider the region of the $\mathbf{l}'$ field bounded by the two forward hypercones from the termini of $d\mathbf{s}$, then all the vectors $d\mathbf{g}$ belonging to this region can be redistributed continuously on one of the hypercones, and just as in § 47 we obtain the vector

$$\frac{d\mathbf{g}}{dS} = l_4' \frac{1 - v\cos\phi}{\sqrt{1 - v^2}}\mathbf{l}'ds.$$

Now we may substitute the value of $\mathbf{l}'$ and obtain

$$d\mathbf{g} = \frac{\epsilon^2}{R^6}\alpha^2(d\mathfrak{S}\times\mathbf{l})^*\mathbf{l}, \tag{113}$$

$$\frac{d\mathbf{g}}{dS} = \frac{\epsilon^2}{R^6}\alpha^2 l_4 \frac{1 - v\cos\phi}{\sqrt{1 - v^2}}\mathbf{l}ds. \tag{114}$$

Before proceeding further with the second of these equations, let us obtain $d\mathbf{g}$ in another form. We may first show that

$$d\mathbf{g} = (d\mathfrak{S}\times\mathbf{l}')^*\mathbf{l}' = (d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}'. \tag{115}$$

For $\mathbf{M}' = \mathbf{u}\times\mathbf{l}'$ where $\mathbf{u}$ is a unit vector perpendicular to $\mathbf{l}'$. Hence

$$(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' = [d\mathfrak{S}^*\cdot(\mathbf{u}\times\mathbf{l}')]\cdot(\mathbf{u}\times\mathbf{l}') = [(d\mathfrak{S}^*\cdot\mathbf{l}')\,\mathbf{u} - (d\mathfrak{S}^*\cdot\mathbf{u})\,\mathbf{l}']\cdot(\mathbf{u}\times\mathbf{l}')$$

by (34). Applying this rule again and noting that $\mathbf{u}\cdot\mathbf{u} = 1$ and $\mathbf{u}\cdot\mathbf{l}' = 0$,

$$(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' = -(d\mathfrak{S}^*\cdot\mathbf{l}')\,\mathbf{l}'.$$

From this, (115) follows by (24). Now we have written $\mathbf{M}'$ as

$$\mathbf{M}' = \mathbf{H}' + \mathbf{E}' = \frac{1}{l_4}\mathbf{l}_s\times\mathbf{e}' - \mathbf{e}'\times\mathbf{k}_4.$$

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[47] Since $\mathbf{l}'$ involves α and therefore $\mathbf{n}\times\mathbf{c}$, the vector $d\mathbf{g}$ is zero for all points in the line of $\mathbf{c}$, and is a maximum when $\mathbf{n}$ is perpendicular to $\mathbf{c}$.

Now we may choose $d\mathfrak{S}$ perpendicular to $\mathbf{k}_4$ and with proper sign, then $d\mathfrak{S}^* = \mathbf{k}_4\, d\mathfrak{S}$. Hence, performing the multiplication,

$$d\mathbf{g} = \left(e'^2 \frac{\mathbf{l}_s}{l_4} + e'^2 \mathbf{k}_4\right) d\mathfrak{S}. \tag{116}$$

Now if $\mathbf{e}'$ is interpreted as electric force in a radiation field, then we are accustomed to regard e'^2 $(= h'^2)$ as the *density of electromagnetic energy,* and the vector $e'^2\, \mathbf{l}_s/l_4$, where $\mathbf{l}_s/l_4$ is a unit vector perpendicular to $\mathbf{e}'$ and $\mathbf{h}'$, as the *Poynting vector.* Therefore $d\mathbf{g}$ becomes a vector of extended momentum of which the components are the total energy and the total momentum in the chosen volume $d\mathfrak{S}$. The vector $d\mathbf{g}$ is moreover independent of any choice of axes and is representative at any point of the tube whose cross section with any chosen space is the volume $d\mathfrak{S}$. But the vector $d\mathbf{g}/d\mathfrak{S}$ obtained by combining the Poynting vector and a vector along the $\mathbf{k}_4$ axis representing the density of energy is by no means independent of the choice of axes. In fact we may state that no way can be found of representing the density of energy by a strictly four dimensional vector. Thus we have a vector of extended momentum for energy-quanta, but not for energy density — an observation which is not without significance in view of certain modern theories of light.

52. It is interesting to note that the same energy vector $d\mathbf{g}$ may be obtained from different 2-vectors $\mathbf{M}'$. For any two singular 2-vectors of the same magnitude and passing through the same element of the hypercone determine the same vector $\mathbf{l}'$ as above defined. If we regard any singular 2-vector $\mathbf{M}'$ produced by an accelerated electron as the extended electromagnetic field of the radiant energy which is moving out along the space projection of the element $\mathbf{l}$ with the velocity of light, then it is evident that, since there is an infinite number of such 2-vectors to which the element $\mathbf{l}$ is common, there is something else necessary to characterize the light besides its energy. In fact a 1-vector such as $\mathbf{l}'$ or $d\mathbf{g}$ upon which the condition is imposed that it shall be singular has three degrees of freedom; a 2-vector such as $\mathbf{M}'$ subject to the two conditions that it shall be singular and uniplanar has four degrees of freedom. It is this additional degree of freedom in $\mathbf{M}'$ which gives rise to such phenomena as polarisation which show a dissymmetry of light with respect to the direction of propagation.

If the vector $d\mathbf{g}$ represents radiant energy (moving out along the hypercone with unity velocity), then the integration of equation (114) around the whole hypercone should give a vector representing the extended momentum of all the energy emitted by the electron, between the ends of the segment $d\mathbf{s}$ of its locus. We wish to evaluate the integral

$$\int \frac{d\mathbf{g}}{dS} dS = ds \int \frac{\epsilon^2}{R^6} \alpha^2 l_4 \frac{1 - v\cos\phi}{\sqrt{1 - v^2}} \mathbf{l} dS. \tag{117}$$

This integration may be simplified by the observation that the vector $d\mathbf{g}$ is not only independent of the direction of the planoid which cuts the boundary of the elementary tube in the surface dS, as has already been shown in general, but is also in this case independent of the position of the planoid, for $d\mathbf{g}/dS$ varies as $1/R^2$ and dS varies as R^2. The integral therefore is the same for any planoid whatsoever, and we may therefore choose for simplicity a planoid perpendicular to the locus $d\mathbf{s}$, and cutting the hypercone in a spherical surface of unit radius, that is $R = l_4 = 1$. Substituting the value of α from (111) gives, since $v = 0$ and $\mathbf{l} = R(\mathbf{w} - \mathbf{n})$,

$$\int \frac{d\mathbf{g}}{dS} dS = ds \int \epsilon^2 (\mathbf{n} \times \mathbf{c})^2 (\mathbf{w} - \mathbf{n})\, d\omega,$$

where $d\omega$ is a solid angle at the center of the sphere subtended by dS. The vector $\mathbf{c}$, normal to $\mathbf{w}$, is then along some diameter of the sphere; and $\mathbf{n}$ is directed from the various points of the surface toward the center. For diametrically opposite points the terms $(\mathbf{c} \times \mathbf{n})^2\, \mathbf{n}$ cancel. We need only integrate the terms $(\mathbf{c} \times \mathbf{n})^2\, \mathbf{w}$. If the diameter determined by $\mathbf{c}$ be taken as polar axis, these terms may be expressed as $c^2 \sin^2\theta\, \mathbf{w}$; and the element of surface is $\sin\theta\, d\theta\, d\phi$. The integral is therefore

$$\int d\mathbf{g} = \frac{8\pi}{3} \epsilon^2 c^2 \mathbf{w} ds. \tag{118}$$

This integral should be the vector of extended momentum for all the energy emitted by the electron between the two points considered, and its projections on any chosen time and space should be the corresponding energy and momentum. If the $\mathbf{k}_4$ axis is chosen parallel to $d\mathbf{s}$, that is if the electron is considered momentarily at rest, we obtain a simple expression; for then $\mathbf{w} = \mathbf{k}_4$, $c^2 = \dot{\mathbf{v}} \cdot \dot{\mathbf{v}}$, and $ds = dt$. The momentum altogether is zero, and the energy is

$$\frac{8\pi}{3} \epsilon^2 (\dot{\mathbf{v}} \cdot \dot{\mathbf{v}})\, dt. \tag{119}$$

When some other $\mathbf{k}_4$ axis is chosen, such that the electron is assumed to have the velocity $\mathbf{v}$, the expression becomes more complicated. Since $\mathbf{w} = (\mathbf{v} + \mathbf{k}_4)/\sqrt{1 - v^2}$ and $ds = \sqrt{1 - v^2}\, dt$, we have from (45),

$$\int d\mathbf{g} = \frac{8\pi}{3} \frac{\epsilon^2}{(1 - v^2)^3} [\dot{\mathbf{v}} \cdot \dot{\mathbf{v}} - (\mathbf{v} \times \dot{\mathbf{v}}) \cdot (\mathbf{v} \times \dot{\mathbf{v}})] (\mathbf{v} + \mathbf{k}_4)\, dt. \qquad (120)$$

The two parts of this expression are precisely in the form obtained by Heaviside and Abraham[48] for the momentum and energy radiated from an accelerated electron.

53. When a singular vector field such as $d\mathbf{g}/dS$ is distributed continuously over a hypercone and is of such a character that its magnitude falls off along any element inversely as the square of the interval of that element measured from the apex (that is, inversely as R^2), or in other words, if it is of such character that the integral of the vector over the surface of intersection of the hypercone with any three dimensional spread is constant, then we may call such a field a simple radiation field. (In three dimensional space the magnitude would fall off inversely with R, and in two dimensional space would be constant.) The fact that the integral of $d\mathbf{g}/dS$ over the intersection of the hypercone with any two parallel planoids is constant may be regarded as equivalent to the law of conservation of radiant energy.

While the discussion which we have given of the vector $d\mathbf{g}$ is in complete accord with current theories of electromagnetic energy, there is another singular 1-vector which is suggested by the geometry and which may be of importance in case it is necessary to revise our ideas of radiant energy. This vector also gives a simple radiation field, in the sense just defined, and is likewise of the second order in $\mathbf{M}'$; but unlike the vectors $d\mathbf{g}$ and $d\mathbf{g}/dS$ it is continuously distributed over a four dimensional field. This is the vector[49] $(\mathbf{w} \cdot \mathbf{M}') \cdot \mathbf{M}' = \mathbf{b}$. The vector $\mathbf{b}$ is along the element of tangency $\mathbf{l}$ by § 39. Indeed if we take $\mathbf{M}'$ from (93) we have

$$\mathbf{b} = (\mathbf{w} \cdot \mathbf{M}') \cdot \mathbf{M}' = \frac{\epsilon^2}{R^2} [\mathbf{c} \cdot \mathbf{c} - (\mathbf{n} \cdot \mathbf{c})^2] (\mathbf{w} - \mathbf{n}) = \frac{\epsilon^2}{R^3} (\mathbf{c} \times \mathbf{n})^2 \mathbf{l}. \qquad (121)$$

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[48] Abraham, Theorie der Elektrizität, **2**, 116.

[49] To obtain a vector, of the second degree in $\mathbf{M}'$, out of $\mathbf{M}'$ itself is out of the question; for the only two products of the second degree in $\mathbf{M}'$ which are geometrically significant, namely $\mathbf{M}' \cdot \mathbf{M}'$ and $\mathbf{M}' \times \mathbf{M}'$, both vanish, since $\mathbf{M}'$ is singular and uniplanar. The vector $\mathbf{b}$ involves not only $\mathbf{M}'$, the field of the electron, but also $\mathbf{w}$ which expresses the state of motion of the electron itself.

If a $\mathbf{k}_4$ axis has been chosen, $\mathbf{b}$ may be obtained in terms of $\mathbf{e}'$, or of $\mathbf{e}'$ and $\mathbf{h}'$. For instance with $\mathbf{M}'$ taken from (108),

$$\mathbf{b} = \left(\frac{\mathbf{v}+\mathbf{k}_4}{\sqrt{1-v^2}}\cdot\frac{\mathbf{l}_s\times\mathbf{e}' + l_4\mathbf{k}_4\times\mathbf{e}'}{l_4}\right)\cdot\frac{\mathbf{l}_s\times\mathbf{e}' + l_4\mathbf{k}_4\times\mathbf{e}'}{l_4}.$$

When we perform the reductions, remembering that $\mathbf{l}_s\cdot\mathbf{e}' = 0$, we find simply

$$\mathbf{b} = \frac{e'^2}{\sqrt{1-v^2}}\left(1-\frac{\mathbf{l}_s\cdot\mathbf{v}}{l_4}\right)\left(\frac{\mathbf{l}_s}{l_4}+\mathbf{k}_4\right). \tag{122}$$

If we use $\mathbf{M}'$ in the form $\mathbf{M}' = \mathbf{E}' + \mathbf{H}'$, we find [50]

$$\mathbf{b} = \frac{1}{\sqrt{1-v^2}}[\overline{\mathbf{e}'\times\mathbf{h}'} + \mathbf{v}\cdot\mathbf{e}'\mathbf{e}' + \mathbf{v}\cdot\mathbf{h}'\mathbf{h}' - h'^2\mathbf{v} + (e'^2 - \mathbf{v}\cdot\overline{\mathbf{e}'\times\mathbf{h}'})\mathbf{k}_4], \tag{123}$$

where $\overline{\mathbf{e}'\times\mathbf{h}'}$ has been used to denote the 1-vector $(\mathbf{e}'\times\mathbf{h}')\cdot\mathbf{k}_{123}$, which is the three dimensional complement of the 2-vector $\mathbf{e}'\times\mathbf{h}'$. Another equivalent form is

$$\mathbf{b} = \frac{l_4 - \mathbf{l}_s\cdot\mathbf{v}}{\sqrt{1-v^2}\,l_4}(\overline{\mathbf{e}'\times\mathbf{h}'} + e'^2\mathbf{k}_4). \tag{124}$$

The coefficient $(l_4 - \mathbf{l}_s\cdot\mathbf{v})/l_4\sqrt{1-v^2}$ is unity when v is negligible compared with the velocity of light, and therefore in all such cases $\mathbf{b}$ is the sum of two vectors one of which is the Poynting vector and the other along $\mathbf{k}_4$ equal in magnitude to the density of energy. Since the vector $\mathbf{b}$ comes so near to being the extended vector of energy density, the possibility is suggested that the energy of an electromagnetic field may not depend solely upon the field itself but to some

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[50] For rapid calculation a rule for obtaining the three dimensional form of some products is useful. The most important of these rules is that if

$$\mathbf{A} = \mathbf{a}\cdot\mathbf{k}_{123} - \mathbf{b}\times\mathbf{k}_4 \quad \text{and} \quad \mathbf{c} = \mathbf{c}_s + c_4\mathbf{k}_4,$$

where $\mathbf{a}$, $\mathbf{b}$ are three dimensional vectors, then

$$\mathbf{c}\cdot\mathbf{A} = \overline{\mathbf{c}_s\times\mathbf{a}} + c_4\mathbf{b} + (\mathbf{c}_s\cdot\mathbf{b})\,\mathbf{k}_4.$$

Thus we have here

$$\begin{aligned}\mathbf{b} = (\mathbf{w}\cdot\mathbf{M}')\cdot\mathbf{M}' &= \frac{1}{\sqrt{1-v^2}}[(\mathbf{v}+\mathbf{k}_4)\cdot(\mathbf{h}'\cdot\mathbf{k}_{123} - \mathbf{e}'\times\mathbf{k}_4)]\cdot(\mathbf{h}'\cdot\mathbf{k}_{123} - \mathbf{e}'\times\mathbf{k}_4)\\ &= \frac{1}{\sqrt{1-v^2}}[\overline{\mathbf{v}\times\mathbf{h}'} + \mathbf{e}' + (\mathbf{v}\cdot\mathbf{e}')\mathbf{k}_4]\cdot(\mathbf{h}'\cdot\mathbf{k}_{123} - \mathbf{e}'\times\mathbf{k}_4)\\ &= \frac{1}{\sqrt{1-v^2}}[\overline{\overline{\mathbf{v}\times\mathbf{h}'}\times\mathbf{h}'} + \overline{\mathbf{e}'\times\mathbf{h}'} + (\mathbf{v}\cdot\mathbf{e}')\mathbf{e}' + (\overline{\mathbf{v}\times\mathbf{h}'}\cdot\mathbf{e}' + \mathbf{e}'\cdot\mathbf{e}')\mathbf{k}_4],\end{aligned}$$

which is identical with the form given.

extent upon the velocity of the emitting electron. It is interesting further to note that by the application of rules already given we may evaluate $\diamondsuit \cdot \mathbf{b}$ and show that it vanishes. Hence

$$\diamondsuit \cdot \mathbf{b} = \nabla \cdot \mathbf{b}_s + \frac{\partial b_4}{\partial t} = 0, \tag{125}$$

where $\mathbf{b}_s$ is the vector which we have just shown to be approximately equal to the Poynting vector, and b_4 is approximately equal to the density of energy. This equation is therefore entirely analogous to the familiar theorem of Poynting. If we integrate over a three-dimensional volume,

$$\int\int\int \nabla \cdot \mathbf{b}_s \, dx_1 dx_2 dx_3 = -\frac{\partial}{\partial t}\int\int\int b_4 dx_1 dx_2 dx_3,$$

or

$$\int\int b_{sn} dS = -\frac{\partial}{\partial t}\int\int\int b_4 dx_1 dx_2 dx_3. \tag{126}$$

Thus the induction of $\mathbf{b}_s$ through any closed surface is equal to the rate of loss of b_4 in the enclosed volume.[51]

If in the vector field $\mathbf{b}$ we cut the hypercone by any planoid, it will be evident that the integral of $\mathbf{b}dS$ over the surface of intersection will be independent of the position and direction of the planoid; for the surface dS always lies in a tangent plane and $\mathbf{b}$ varies inversely as R^2 and hence as dS. The vector $\mathbf{b}dS$ bears a simple relation to $d\mathbf{g}$ which we have studied. For $d\mathbf{g} = (d\mathfrak{S}^* \cdot \mathbf{M}') \cdot \mathbf{M}'$, where $d\mathfrak{S}$ is determined by any planoid. We may therefore choose $d\mathfrak{S}$ perpendicular to $d\mathbf{s}$, that is, to $\mathbf{w}$. Then $d\mathfrak{S}^*$ is $\mathbf{w}\,d\mathfrak{S}$ and $d\mathfrak{S} = dSds$, and since $\mathbf{b}$ by definition is $(\mathbf{w} \cdot \mathbf{M}') \cdot \mathbf{M}'$, the integral of $d\mathbf{g}$ is the product of ds and the integral of $\mathbf{b}dS$. We might therefore by a consideration of $\mathbf{b}$ alone have obtained the same vector of extended momentum for the total energy emitted by an electron in the interval $d\mathbf{s}$.

We shall not pursue further the study of this interesting vector $\mathbf{b}$, but it may be well to point out that the two fields $\mathbf{M}'$ and $\mathbf{b}$ cannot both be additive. For since $\mathbf{b}$ is quadratic in $\mathbf{M}'$, we obtain a different result when we obtain $\mathbf{b}$ from a resultant $\mathbf{M}$ (no longer necessarily a singular vector) and when we add the $\mathbf{b}$'s obtained from the original $\mathbf{M}''$s. All the classic ideas of electromagnetic energy assume that it is the vectors $\mathbf{M}$ that are additive at a point.

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[51] In general if a 1-vector field in four dimensions is of such a character that its four-dimensional divergence vanishes, we may obtain in three dimensions an equation of the type just found, wherein the surface integral over a closed surface of the space component of the vector is equal to the negative time derivative of the integral of the time-component of the vector over the enclosed volume. Such an equation may be interpreted as a continuity or conservation equation whenever the space component appears as a velocity multiplied by the quantity defined by the time-component.

The Field of Continuous Distributions of Electricity.

54. Since the locus of an electric charge is not a singular line, we may regard the charge as distributed continuously over a given region or regions rather than as concentrated at one or more discrete points. Thus instead of a single vector representing the locus of an electron, we may consider a vector field. Let a small (δ)-tube be parallel to and comprise n electron-loci each of charge ϵ. Then we may replace these on the one hand by a single vector $n\epsilon\mathbf{w}$, and on the other hand by a vector field $\mathbf{q}$ such that, if $d\mathfrak{S}$ is the volume of any portion of the tube cut off by a planoid perpendicular to $\mathbf{w}$,

$$\int \mathbf{q}\, d\mathfrak{S} = n\epsilon\mathbf{w}.$$

Or if $d\boldsymbol{\mathfrak{S}}$ is the vector volume cut off by any planoid whatever, then as in § 45,

$$\int (d\boldsymbol{\mathfrak{S}}\times\mathbf{q})^{*}\mathbf{w} = n\epsilon\mathbf{w}. \tag{127}$$

If now we write

$$\mathbf{q} = \rho_0\mathbf{w}, \tag{128}$$

ρ_0 evidently represents the density of electricity as it appears to an observer stationary with respect to the charge. To an observer with respect to whom the charge appears to be moving with the velocity $\mathbf{v}$ the density appears to be different. For we may write (127) in the form

$$\int - (d\boldsymbol{\mathfrak{S}}^{*}\cdot\mathbf{q})\ \mathbf{w};$$

and if $d\boldsymbol{\mathfrak{S}}$ is the volume cut off by the planoid perpendicular to the chosen time-axis $\mathbf{k}_4$, $d\boldsymbol{\mathfrak{S}}^{*} = d\mathfrak{S}\mathbf{k}_4$; then writing

$$\mathbf{w} = (\mathbf{v} + \mathbf{k}_4)/\sqrt{1 - v^2},$$

we have

$$\int \frac{\rho_0\mathbf{w}}{\sqrt{1 - v^2}}\, d\mathfrak{S} = n\epsilon\mathbf{w}. \tag{129}$$

If then ρ is the density of the moving charge, we must write

$$\rho = \frac{\rho_0}{\sqrt{1 - v^2}}. \tag{130}$$

When we compare the two vectors

$$\epsilon\mathbf{w} = \frac{\epsilon}{\sqrt{1 - v^2}}(\mathbf{v} + \mathbf{k}_4) \qquad \text{and} \qquad \rho_0\mathbf{w} = \rho\,(\mathbf{v} + \mathbf{k}_4)$$

with the two vectors which we have obtained for a material system

$$m_0\mathbf{w} = m\,(\mathbf{v} + \mathbf{k}_4) \qquad \text{and} \qquad \mu_0\mathbf{w} = \frac{\mu_0}{\sqrt{1 - v^2}}(\mathbf{v} + \mathbf{k}_4)$$

we see that they are identical in mathematical form. But the components of $\epsilon\mathbf{w}$ are not quantities which are commonly used in physics, while the components of $\rho_0\mathbf{w}$ are the density of electricity and of electric current. On the other hand the components of $m_0\mathbf{w}$ are the fundamental quantities known as mass and momentum, while the components of $\mu_0\mathbf{w}$ are not commonly used. This is probably due to the fact that the fundamental conservation law for electricity is $\Sigma\epsilon =$ const., whereas the fundamental conservation law for mass is not $\Sigma m_0 =$ const., but $\Sigma m =$ const.

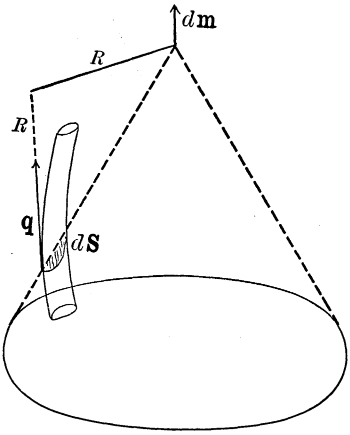

FIGURE 28.

55. We may now construct the potential at a point due to a continuous distribution of electricity, directly from (91) and (127).

$$\mathbf{m} = \int (d\mathfrak{S}\times\mathbf{q})^* \frac{\mathbf{w}}{R}. \tag{131}$$

The interpretation of this equation will be evident from an examination of a diagram which is an immediate extension of the one previously used in discussing potentials. And we may then show that, when a particular space and time have been assumed, the components of the vector $\mathbf{m}$ on the chosen space and time are the ordinary "retarded" potentials.

If (Figure 28)[52] we draw the backward hypercone from the point at which the potential $\mathbf{m}$ is to be determined, and if this backward hypercone cuts an elementary tube of the field $\mathbf{q}$ in the vector volume $d\boldsymbol{\mathfrak{S}}$, then R is the perpendicular interval from the point in question to $\mathbf{w}$ or $\mathbf{w}$ produced (where $\mathbf{w}$ is the direction of $\mathbf{q}$ at the point where the tube cuts the cone). That part of the additive potential vector $\mathbf{m}$ which is due to this particular tube is

$$d\mathbf{m} = (d\boldsymbol{\mathfrak{S}}\times\mathbf{q})^*\frac{\mathbf{w}}{R} = -\,d\boldsymbol{\mathfrak{S}}^*\cdot\mathbf{q}\frac{\mathbf{w}}{R}. \qquad (132)$$

Evidently the integration of $d\mathbf{m}$ is to be taken over the whole three dimensional spread produced by the intersection of the backward hypercone with the whole assemblage of infinitesimal tubes.

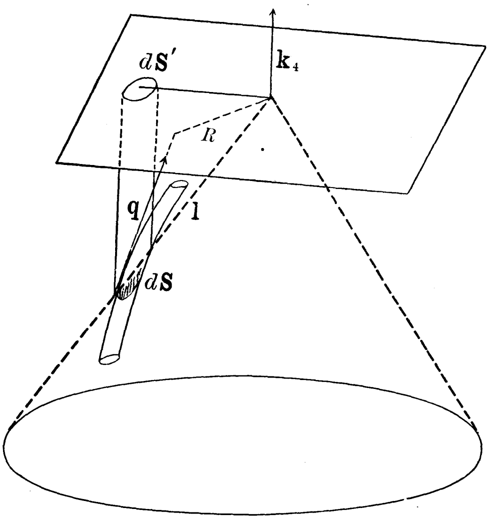

FIGURE 29.

Now if (Figure 29) we construct any planoid through the point in question, the retarded potentials are calculated as follows. This planoid, which we may regard as our space, is divided into elements of volume $d\mathfrak{S}'$ (corresponding to dS' in the figure). We consider the values $[\rho]$ and $[\rho\mathbf{v}]$ of the density and the current density which were in the element $d\mathfrak{S}'$ at a time previous by the length of time required for light to pass from $d\mathfrak{S}'$ to the point in question. From the four dimensional point of view this means that we project the element $d\mathfrak{S}'$ parallel to the time-axis upon the hypercone, and take as $[\rho]$ and $[\rho\mathbf{v}]$ the projections on time and space of the vector $\mathbf{q}$ at this point of the hypercone. We then form the integrals

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[52] Figure 28 and Figure 29 are drawn and lettered for one dimension lower.

$$\int \frac{[\rho]}{r}\, d\mathfrak{S}' \quad \text{and} \quad \int \frac{[\rho\mathbf{v}]}{r}\, d\mathfrak{S}', \tag{133}$$

where r is the distance from $d\mathfrak{S}'$ to the point at which the potential is wanted.

Let us now consider the element $d\mathbf{m}$ of our potential. The vector $d\boldsymbol{\mathfrak{S}}$ (corresponding to $d\mathbf{S}$ of the figure), being cut out of the hypercone, is a singular 3-vector, and its complement $d\boldsymbol{\mathfrak{S}}^*$ is therefore a singular 1-vector. Hence $d\mathfrak{S}'$ is numerically the projection of $d\boldsymbol{\mathfrak{S}}^*$ upon $\mathbf{k}_4$, and it is readily seen that

$$\frac{d\boldsymbol{\mathfrak{S}}^*}{d\mathfrak{S}'} = \frac{\mathbf{1}}{l_4}.$$

Substituting in (132),

$$d\mathbf{m} = -\frac{\mathbf{1}\cdot\mathbf{q}}{l_4}\frac{\mathbf{w}}{R}\, d\mathfrak{S}' = \frac{\mathbf{1}\cdot\rho_0\mathbf{w}}{l_4}\frac{\mathbf{w}}{R}\, d\mathfrak{S}'.$$

But $\mathbf{1}\cdot\mathbf{w} = -R$ by (80) and l_4 is equal to l_3, that is, to the r in (133). Hence

$$\mathbf{m} = \int \frac{[\rho\mathbf{v}] + [\rho]\,\mathbf{k}_4}{r}\, d\mathfrak{S}'. \tag{134}$$

If we designate the vector and scalar potentials as $\mathbf{a}$ and ϕ respectively, then

$$\mathbf{m} = \mathbf{a} + \phi\mathbf{k}_4. \tag{135}$$

We may show as before [53] that

$$\Diamond\cdot\mathbf{m} = 0 \quad \text{or} \quad \nabla\cdot\mathbf{a} + \frac{\partial\phi}{\partial t} = 0. \tag{136}$$

We have seen (§ 44) that $\Diamond\cdot\Diamond\,\mathbf{p} = 0$, or $\Diamond^2\mathbf{p} = 0$, and consequently $\Diamond^2\mathbf{m} = 0$ in the case of a point electron for all points not upon the locus of the electron. In the case of a continuous distribution of electricity we have [54]

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[53] A single differentiation under the sign of integration is permissible if ρ_0 remains finite; but a second differentiation is not permissible, as is well known in the theory of the potential.

$$\Diamond^2\mathbf{m} = -4\pi\mathbf{q}, \tag{137}$$

which might be proved directly; but this is unnecessary since it has frequently been shown by familiar methods that

$$\Diamond^2\mathbf{a} = -4\pi\rho\mathbf{v} \qquad \text{and} \qquad \Diamond^2\phi = -4\pi\rho. \tag{138}$$

Furthermore it is unnecessary to evaluate once more in detail the 2-vector

$$\mathbf{M} = \Diamond\times\mathbf{m} = \nabla\times\mathbf{a} + \left(\nabla\phi + \frac{\partial\mathbf{a}}{\partial t}\right)\times\mathbf{k}_4. \tag{139}$$

For $\nabla\times\mathbf{a}$ is the three dimensional complement of what is ordinarily known as curl $\mathbf{a}$ or $\mathbf{h}$, and $\nabla\phi + \dot{\mathbf{a}} = -\mathbf{e}$. Hence

$$\mathbf{M} = \mathbf{H} + \dot{\mathbf{E}},$$

where the components of $\mathbf{H}$ and $\mathbf{E}$ are once more the components of magnetic and electric force.

56. Whether the 2-vector $\mathbf{M}$ of extended electric and magnetic force be derived from a number of point charges or from a charge continuously distributed, it is in general a complex or biplanar 2-vector.[55] The two invariants of $\mathbf{M}$ are $\mathbf{M}\cdot\mathbf{M}$ and $\mathbf{M}\cdot\mathbf{M}^* = (\mathbf{M}\times\mathbf{M})^*$. If, after choosing space and time axes, we write

$$\begin{aligned} \mathbf{M} &= h_1\mathbf{k}_{23} + h_2\mathbf{k}_{31} + h_3\mathbf{k}_{12} - e_1\mathbf{k}_{14} - e_2\mathbf{k}_{24} - e_3\mathbf{k}_{34}, \\ \mathbf{M}^* &= e_1\mathbf{k}_{23} + e_2\mathbf{k}_{31} + e_3\mathbf{k}_{12} + h_1\mathbf{k}_{14} + h_2\mathbf{k}_{24} + h_3\mathbf{k}_{34}, \end{aligned} \tag{140}$$

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[54] The vector $4\pi\mathbf{q}$ which we use is identical with the vector $\mathbf{q}$ used by Lewis, owing to a different choice of units of electrical quantity.

[55] Since it is customary to divide a complex 2-vector into the two completely perpendicular uniplanar vectors which are uniquely determined, one being a (γ)-vector, the other a (δ)-vector, we might expect that the two lines of intersection of the (δ)-plane with the hypercone, and their projections upon a chosen space, might prove important. This is, however, not the case, although indeed from an analytic point of view the four directions, two of them imaginary, in which the hypercone is cut by the completely perpendicular (δ)-vector and (γ)-vector form a set of four independent directions possessing some advantages over the system $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$. In fact four vectors $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$, $\mathbf{j}_4$ can be selected along these directions such that

$$\mathbf{j}_1\cdot\mathbf{j}_1 = \mathbf{j}_2\cdot\mathbf{j}_2 = \mathbf{j}_3\cdot\mathbf{j}_3 = \mathbf{j}_4\cdot\mathbf{j}_4 = 0, \qquad \mathbf{j}_1\cdot\mathbf{j}_2 = \mathbf{j}_3\cdot\mathbf{j}_4 = 1,$$
$$\mathbf{j}_1\cdot\mathbf{j}_3 = \mathbf{j}_1\cdot\mathbf{j}_4 = \mathbf{j}_2\cdot\mathbf{j}_3 = \mathbf{j}_2\cdot\mathbf{j}_4 = 0.$$

In terms of such a set of vectors the differential of arc is given by the equation

$$d\mathbf{r}\cdot d\mathbf{r} = dx^2 + dy^2 + dz^2 - dt^2 = Adudv + Bdwds.$$

(See Bateman, Proc. Lond. Math. Soc. [2] **10**, 107).

Other vectors which might be thought important would be the two lines in which the completely perpendicular planes cut the planoid which is taken then $\mathbf{M}\cdot\mathbf{M} = h^2 - e^2 = 2L$, where L is known as the Lagrangian function, and $\mathbf{M}\cdot\mathbf{M}^* = 2\,\mathbf{e}\cdot\mathbf{h}$. It is not surprising that the Lagrangian function should prove to be one of the fundamental invariants, but it is strange that the other invariant should be a quantity which has not been regarded as of fundamental importance in electromagnetic theory.

Since we have obtained our 2-vector from the equation

$$\mathbf{M} = \Diamond \times \mathbf{m},$$

we may readily evaluate $\Diamond \times \mathbf{M}$ and $\Diamond \cdot \mathbf{M}$. By (51) as a mathematical identity we have

$$\Diamond \times \mathbf{M} = \Diamond \times \Diamond \times \mathbf{m} = 0. \tag{141}$$

By (55)

$$\Diamond \cdot \mathbf{M} = \Diamond \cdot (\Diamond \times \mathbf{m}) = \Diamond(\Diamond \cdot \mathbf{m}) - (\Diamond \cdot \Diamond)\mathbf{m};$$

and since we have seen that in general $\Diamond \cdot \mathbf{m} = 0$, and substituting for $\Diamond \cdot \Diamond \mathbf{m}$ or $\Diamond^2 \mathbf{m}$ from the preceding section,[56] we find

$$\Diamond \cdot \mathbf{M} = 4\pi \mathbf{q}. \tag{142}$$

By (52) as a mathematical identity,

$$\Diamond \cdot (\Diamond \cdot \mathbf{M}) = 0. \tag{143}$$

By the expansion of these equations we obtain directly the familiar equations of the electromagnetic field and the continuity equation

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as space. Following the method of § 38 we may write $\mathbf{M}$ as the sum of its two completely perpendicular parts in the form

$$\mathbf{M} = \frac{1}{2}\,\frac{\left(\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2} + \mathbf{M}\cdot\mathbf{M}\right)\mathbf{M} + (\mathbf{M}\cdot\mathbf{M}^*)\,\mathbf{M}^*}{\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2}}$$
$$+ \frac{1}{2}\,\frac{\left(\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2} - \mathbf{M}\cdot\mathbf{M}\right)\mathbf{M} - (\mathbf{M}\cdot\mathbf{M}^*)\,\mathbf{M}^*}{\sqrt{(\mathbf{M}\cdot\mathbf{M})^2 + (\mathbf{M}\cdot\mathbf{M}^*)^2}}.$$

Now the lines in which these two completely perpendicular planes cut the space $\mathbf{k}_{123}$ may be found by multiplying the planes by $\mathbf{k}_4$ by inner multiplication. As $\mathbf{k}_4\cdot\mathbf{M} = \mathbf{e}$ and $\mathbf{k}_4\cdot\mathbf{M}^* = -\,\mathbf{h}$, we have for the lines

$$\frac{1}{2}\,\frac{\left(\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2} + L\right)\mathbf{e} - (\mathbf{e}\cdot\mathbf{h})\mathbf{h}}{\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2}}, \qquad \frac{1}{2}\,\frac{\left(\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2} - L\right)\mathbf{e} + (\mathbf{e}\cdot\mathbf{h})\mathbf{h}}{\sqrt{L^2 + (\mathbf{e}\cdot\mathbf{h})^2}}.$$

These vectors, however, like those mentioned above, are not found to be important in electromagnetic theory.

[56] cf. equation (85).

expressing the conservation of electricity. We may write (141) in the form $\Diamond\cdot\mathbf{M}^* = 0$. Expressing $\mathbf{M}^*$ as in (140), this equation becomes

$$\left.\begin{aligned}\overline{\nabla\times\mathbf{e}} + \frac{\partial\mathbf{h}}{\partial t} &= 0,\\ \nabla\cdot\mathbf{h} &= 0.\end{aligned}\right\}$$

Similarly from (142)

$$\left.\begin{aligned}\overline{\nabla\times\mathbf{h}} - \frac{\partial\mathbf{e}}{\partial t} &= 4\,\pi\rho\mathbf{v},\\ \nabla\cdot\mathbf{e} &= 4\,\pi\rho.\end{aligned}\right\}$$

These are the well known field equations. Finally (143) gives the continuity equation

$$\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0.$$

It cannot be too strongly emphasized that all these equations follow from the theorems of our four dimensional geometry without any further assumption than that the geometrical vector potential field derived from the locus of an electric charge is the extended electromagnetic vector potential.

57. We have seen that the singular 2-vector field $\mathbf{M}'$ produced by an accelerated electron determines a vector $d\mathbf{g}$ of four dimensional significance involving quantities which may be identified with energy and momentum in the radiation field. A search for similar vectors due to the field $\mathbf{M}$, which in general is not singular, proves, however, to be unsuccessful. In the case of radiation we wrote

$$d\mathbf{g} = (d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}',$$

or since it is readily shown (see footnote, § 62) that in this case $(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' = (d\mathfrak{S}^*\cdot\mathbf{M}'^*)\cdot\mathbf{M}'^*$ we could have obtained a more symmetrical form

$$d\mathbf{g} = \tfrac{1}{2}[(d\mathfrak{S}^*\cdot\mathbf{M}')\cdot\mathbf{M}' + (d\mathfrak{S}^*\cdot\mathbf{M}'^*)\cdot\mathbf{M}'^*]. \tag{144}$$

In the case of the vector $\mathbf{M}$ we may write by analogy

$$\tfrac{1}{2}[(d\mathfrak{S}^*\cdot\mathbf{M})\cdot\mathbf{M} + (d\mathfrak{S}^*\cdot\mathbf{M}^*)\cdot\mathbf{M}^*], \tag{145}$$

where $d\mathfrak{S}$ is the vector volume produced by intersecting a selected portion of the four dimensional field by a planoid. However, this cannot be made to give rise to a real vector in a four dimensional sense, but will only have meaning for the particular planoid chosen.

If we choose a particular $\mathbf{k}_4$ axis and its perpendicular planoid, then $d\mathfrak{S}^* = d\mathfrak{S}\ \mathbf{k}_4$ and the above expression becomes

$$\tfrac{1}{2}[(\mathbf{k}_4\cdot\mathbf{M})\cdot\mathbf{M} + (\mathbf{k}_4\cdot\mathbf{M}^*)\cdot\mathbf{M}^*]\,d\mathfrak{S}. \tag{146}$$

We may perform the operations here indicated upon the expanded form (140) of $\mathbf{M}$ and obtain[57]

$$[\overline{\mathbf{e}\times\mathbf{h}} + \tfrac{1}{2}(e^2 + h^2)\,\mathbf{k}_4]\,d\mathfrak{S}. \tag{147}$$

Now $\overline{\mathbf{e}\times\mathbf{h}}$, the complement in three dimensional space of $\mathbf{e}\times\mathbf{h}$, and $\frac{1}{2}(e^2 + h^2)$ are the familiar expressions for the Poynting vector and the density of electromagnetic energy, and the above expression therefore represents what is ordinarily regarded as the total electromagnetic momentum and energy in the volume $d\mathfrak{S}$.

Now after the axes have been chosen we may perform similar operations with $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$. Thus

$$\begin{aligned}
\tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= X_x\mathbf{k}_1 + X_y\mathbf{k}_2 + X_z\mathbf{k}_3 - X_t\mathbf{k}_4,\\
\tfrac{1}{2}(\mathbf{k}_2\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_2\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= Y_x\mathbf{k}_1 + Y_y\mathbf{k}_2 + Y_z\mathbf{k}_3 - Y_t\mathbf{k}_4,\\
\tfrac{1}{2}(\mathbf{k}_3\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_3\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= Z_x\mathbf{k}_1 + Z_y\mathbf{k}_2 + Z_z\mathbf{k}_3 - Z_t\mathbf{k}_4,\\
\tfrac{1}{2}(\mathbf{k}_4\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_4\cdot\mathbf{M}^*)\cdot\mathbf{M}^* &= T_x\mathbf{k}_1 + T_y\mathbf{k}_2 + T_z\mathbf{k}_3 + T_t\mathbf{k}_4,
\end{aligned}$$

where

$$\begin{aligned}
X_x &= \tfrac{1}{2}(e_1^2 - e_2^2 - e_3^2 + h_1^2 - h_2^2 - h_3^2),\\
Y_y &= \tfrac{1}{2}(e_2^2 - e_3^2 - e_1^2 + h_2^2 - h_3^2 - h_1^2),\\
Z_z &= \tfrac{1}{2}(e_3^2 - e_1^2 - e_2^2 + h_3^2 - h_1^2 - h_2^2),\\
T_t &= \tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2 + h_1^2 + h_2^2 + h_3^2),\\
X_y &= Y_x = e_1e_2 + h_1h_2, \text{ etc.},\\
T_x &= X_t = e_2h_3 - e_3h_2, \text{ etc.}
\end{aligned}$$

In these equations X_x, etc., are the familiar expressions for the components of the Maxwell strains; T_x, T_y, T_z are the components of the Poynting vector; and T_t is that which is ordinarily assumed to be the density of electromagnetic energy. This procedure is essentially that of Minkowski. We may reproduce his procedure exactly with the aid of dyadics. It may readily be shown (see appendix, § 62) that if $\mathbf{M}$ is any 2-vector, and I the unit dyadic or idemfactor, then the dyadics

$$\Phi = (\mathrm{I}\cdot\mathbf{M})\cdot(\mathrm{I}\cdot\mathbf{M}) \qquad \Phi^* = (\mathrm{I}\cdot\mathbf{M}^*)\cdot(\mathrm{I}\cdot\mathbf{M}^*)$$

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[57] For abbreviated methods see a footnote in § 53.

are such that

$$\mathbf{a}\cdot\Phi = (\mathbf{a}\cdot\mathbf{M})\cdot\mathbf{M} \qquad \mathbf{a}\cdot\Phi^* = (\mathbf{a}\cdot\mathbf{M}^*)\cdot\mathbf{M}^*,$$

where **a** is any 1-vector. The expressions which we obtained from **M** and $\mathbf{k}_1, \mathbf{k}_2, \ldots$ in the form

$$\tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M})\cdot\mathbf{M} + \tfrac{1}{2}(\mathbf{k}_1\cdot\mathbf{M}^*)\cdot\mathbf{M}^*, \text{ etc.}$$

might therefore equally well have been written

$$\tfrac{1}{2}\mathbf{k}_1\cdot(\Phi + \Phi^*), \text{ etc.}$$

It is these latter expressions which Minkowski obtained. The dyadic $\frac{1}{2}(\Phi + \Phi^*)$ is identical with Minkowski's matrix S, except in as far as he used imaginary space, and distinguished between electric force and displacement and between magnetic force and induction.[58]

While, as we see, the use of the dyadic $\frac{1}{2}(\Phi + \Phi^*)$ yields no results which are not also obtainable by the methods of simple vector analysis, yet to one who is familiar with the dyadic method it frequently affords a considerable gain in simplicity. Thus for example we may obtain an important result by considering the expression $\frac{1}{2}\Diamond\cdot(\Phi + \Phi^*)$, which may be shown to vanish in free space.[59] Now, if Ψ_s be the three dimensional dyadic of the Maxwell strains, if $\overline{\mathbf{e}\times\mathbf{h}}$ is the Poynting vector, and if T_t is the density of energy, we have

$$0 = \tfrac{1}{2}\Diamond\cdot(\Phi + \Phi^*) = \Diamond\cdot(\Psi_s - \overline{\mathbf{e}\times\mathbf{h}}\,\mathbf{k}_4 - \mathbf{k}_4\overline{\mathbf{e}\times\mathbf{h}} - \mathbf{k}_4\mathbf{k}_4 T_t), \quad (148)$$

or

$$\nabla\cdot\Psi_s - \frac{\partial}{\partial t}(\overline{\mathbf{e}\times\mathbf{h}}) = 0 \qquad \text{and} \qquad \nabla\cdot\overline{\mathbf{e}\times\mathbf{h}} + \frac{\partial}{\partial t}T_t = 0. \quad (149)$$

The first is the important equation of Lorentz connecting the force

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[58] The form of the dyadic $\Psi = \frac{1}{2}(\Phi + \Phi^*)$ is

$$\begin{aligned} &X_x\mathbf{k}_1\mathbf{k}_1 + X_y\mathbf{k}_1\mathbf{k}_2 + X_z\mathbf{k}_1\mathbf{k}_3 - X_t\mathbf{k}_1\mathbf{k}_4 \\ &+ Y_x\mathbf{k}_2\mathbf{k}_1 + Y_y\mathbf{k}_2\mathbf{k}_2 + Y_z\mathbf{k}_2\mathbf{k}_3 - Y_t\mathbf{k}_2\mathbf{k}_4 \\ &+ Z_x\mathbf{k}_3\mathbf{k}_1 + Z_y\mathbf{k}_3\mathbf{k}_2 + Z_z\mathbf{k}_3\mathbf{k}_3 - Z_t\mathbf{k}_3\mathbf{k}_4 \\ &- T_x\mathbf{k}_4\mathbf{k}_1 - T_y\mathbf{k}_4\mathbf{k}_2 - T_z\mathbf{k}_4\mathbf{k}_3 - T_t\mathbf{k}_4\mathbf{k}_4. \end{aligned}$$

[59] From (158), with $\mathbf{A} = \mathbf{M}$, $\mathbf{A}' = \mathbf{M}$, and from (141) and (142), since in free space $\mathbf{q} = 0$. Where there is electricity the equation would be

$$\tfrac{1}{2}\Diamond\cdot\Psi = 4\pi\mathbf{q}\cdot\mathbf{M}.$$

due to the Maxwell strains and the rate of change of the Poynting vector; the second is Poynting's theorem.[60]

Mechanics of a Material System, and Gravitation.

58. The mechanics of a particle which we have treated in restricted cases in § 21 and § 36 can now be completely generalized. If m_0 is the mass of a particle, and $\mathbf{w}$ the unit tangent to its locus, then

$$m_0\mathbf{w} = m(\mathbf{v} + \mathbf{k}_4)$$

is the vector of extended momentum, whose projections on any chosen space and time are $m\mathbf{v}$, the momentum, and m, the mass or energy. If we consider any number of such vectors, we may state the laws of conservation of momentum, mass and energy in a single theorem as follows. *The sum of all the vectors of extended momentum is constant,* that is, the sum of all such vectors cutting any unclosed and continuous three dimensional (γ)-spread is independent of the (γ)-spread chosen. This law is, however, true only when we state that wherever there is energy there is a vector of extended momentum, whether or not this energy is associated with that which is ordinarily known as a material system. Thus in § 51 we have discussed the vector $d\mathbf{g}$ which we have identified with the vector of extended momentum of radiant electromagnetic energy. A *Hohlraum* obeys all the laws of a material system, and must be treated as such. We shall mention presently another form of radiant energy to which also we must assign an extended momentum.

Just as the discrete locus of an electric charge was replaced by a continuously distributed field of density vectors, we might regard a material system as a continuum. Thus if we have a small (δ)-tube parallel to and comprising one or more (δ)-lines of which the resultant vector is $m_0\mathbf{w}$, we may replace this vector by the expression $(d\boldsymbol{\mathfrak{S}}\times\mu_0\mathbf{w})^*\mathbf{w}$, where $d\boldsymbol{\mathfrak{S}}$ is the intersection of the tube with any planoid, and $\mu_0\mathbf{w}$ is the vector of the distributed field. If $d\boldsymbol{\mathfrak{S}}$ is taken perpendicular to $\mathbf{w}$, this reduces to $\mu_0\mathbf{w}d\mathfrak{S}$, and therefore μ_0 is the density as it appears to an observer at rest with respect to the system. It must, however,

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[60] In case there is electricity present, these equations become respectively

$$\nabla\bullet\Psi_s - \frac{\partial}{\partial t}\overline{\mathbf{e}\times\mathbf{h}} = 4\pi\rho(\mathbf{e} + \overline{\mathbf{v}\times\mathbf{h}}), \qquad \nabla\bullet\overline{\mathbf{e}\times\mathbf{h}} + \frac{\partial}{\partial t}T_t = -4\pi\rho\mathbf{v}\bullet\mathbf{e}.$$

Note that if $\mathbf{v}$ is small, the second equation is corrected by the small term $-4\pi\rho\mathbf{v}\bullet\mathbf{e}$, whereas the first has the large correction $4\pi\rho(\mathbf{e} + \overline{\mathbf{v}\times\mathbf{h}})$, approximately $4\pi\rho\mathbf{e}$.

be borne in mind that when the system in question embraces any energy which is moving with the velocity of light, this method fails completely. And this is an essential difference between a system of electric charges and a system of matter or energy. Indeed a consideration of the properties of a *Hohlraum* shows that it may be unsafe in any case to assume that a material system is not composed wholly or in part of energy moving with the velocity of light.

59. In the study of hydrodynamics cases are considered in which the different portions of the fluid exert forces upon one another, and these forces may be themselves due to a flow of energy with the velocity of light. In fact it is only when we consider a fluid devoid of such mutual forces that we are able to obtain from our continuously distributed field and the law of extended momentum the known equation of hydrodynamics. Let us consider a continuously distributed field divided into infinitesimal tubes in each of which the extended momentum is now written as $(d\mathfrak{S}\times\mu_0\mathbf{w})^*\mathbf{w}$. Then our conservation law leads to the equation

$$\int(d\mathfrak{S}\times\mu_0\mathbf{w})^*\mathbf{w} = \mathbf{const.} \tag{150}$$

Or if we consider a portion of the field composed of a number of adjoining tubes and cut off by two different planoids, then since none of the vectors of extended momentum cut the boundary tube the integral of our vector over the whole three dimensional boundary of this four dimensional region is merely the integral over the two planoids namely,

$$-\int(d\mathfrak{S}^*\cdot\mu_0\mathbf{w})\mathbf{w} = 0 = -\int d\mathfrak{S}^*\cdot(\mu_0\mathbf{ww}),$$

by definition of the dyadic $\mu_0\mathbf{ww}$. Now by the application of (65) we may convert this triple integral into a quadruple integral. Thus

$$\int d^*\mathfrak{S}\cdot(\mu_0\mathbf{ww}) = \int d\Sigma^*\Diamond\cdot(\mu_0\mathbf{ww}) = 0.$$

Hence

$$\Diamond\cdot(\mu_0\mathbf{ww}) = 0. \tag{151}$$

If now we set $\mathbf{w} = (\mathbf{v} + \mathbf{k}_4)/\sqrt{1 - v^2}$ and $\mu = \mu_0/(1 - v^2)$ by (88), this gives by expansion[61]

$$\Diamond\cdot[\mu(\mathbf{v} + \mathbf{k}_4)(\mathbf{v} + \mathbf{k}_4)] = [\Diamond\cdot\mu(\mathbf{v} + \mathbf{k}_4)](\mathbf{v} + \mathbf{k}_4) + [\mu(\mathbf{v} + \mathbf{k}_4)\cdot\Diamond](\mathbf{v} + \mathbf{k}_4) = 0,$$

---

[61] If $\mathbf{ab}$ is a dyadic, evidently $\Diamond\cdot(\mathbf{ab}) = (\Diamond\cdot\mathbf{a})\mathbf{b} + (\mathbf{a}\cdot\Diamond)\mathbf{b}$.

or

$$\left[\nabla\cdot(\mu\mathbf{v}) + \frac{\partial\mu}{\partial t}\right](\mathbf{v} + \mathbf{k}_4) + \mu\left[(\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t}\right] = 0,$$

Hence the space and time components both vanish, and

$$\nabla\cdot(\mu\mathbf{v}) + \frac{\partial\mu}{\partial t} = 0, \tag{152}$$

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t} = 0. \tag{153}$$

The first of these two is the continuity equation, the second is the dynamical equation of hydrodynamics in the present restricted case.[62] The fact that we are thus led not to the general laws of hydrodynamics but merely to the laws for a comparatively trivial case shows the inadequacy of any attempt to distribute the vectors of extended momentum into a continuous field.

Minkowski added to his great memoir on the "Grundgleichungen für die electromagnetischen Vorgänge" an appendix on mechanics which seems to have been more hastily written. In this section he bases his analysis upon two assumptions which must be considered as fundamentally erroneous. The first of these is that $\mu = \mu_0/\sqrt{1 - v^2}$; and the second that Σm_0 is a constant.[63] The results should be that $\mu = \mu_0/(1 - v^2)$ and that Σm is a constant. We have already discussed (§ 23) cases in which m_0 is not a constant.

60. Every locus of a particle to which belongs the vector $m_0\mathbf{w}$ gives rise to the geometric vector fields

$$m_0\mathbf{p} = m_0\mathbf{w}/R \qquad \text{and} \qquad m_0\mathbf{P} = m_0\Diamond\times\mathbf{p}.$$

By replacing the constant ϵ by the constant m_0 we might proceed to reproduce identically all of the formulas which we have obtained for the electromagnetic field. If a suitable unit of mass be chosen, we should then observe that in case axes are so taken that the particle appears at rest, the space vector $m_0\mathbf{P}\cdot\mathbf{k}_4$ becomes identical in form with gravitational force, and the time component of $m_0\mathbf{p}$ with gravitational potential. When the particle is not at rest it is evident that just as in electromagnetics we must add to the scalar potential a vector potential, and to the (corrected) gravitational force another force which by analogy we may call gravito-magnetic. In every other respect, moreover, the two problems must be completely analogous. Thus an accelerated particle must give rise to a singular vector field which we should expect to be associated with the flow of a new form of radiant energy.[64]

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[62] It may well be that the introduction of additional terms sufficient to give (153) a form as general as that ordinarily used in hydrodynamics would not require serious modifications in (152). For in ordinary units the pressure of light is measured by the density of electromagnetic energy, whereas the mass of the light is its energy divided by the square of the velocity of light. Compare also the fact that the changes in the equations (149) when electricity is present is small in one case and large in the other.

[63] The second of these errors has already been pointed out by Abraham, Rend. Circ. Mat. di Palermo, **30**, 45.

APPENDIX.

Dyadics.

61. The dyad or formal product of vectors, introduced in 1844 by Grassmann under the name of open product, was given a fundamental position in vector analysis by Gibbs. Gibbs also developed the idea of the dyadic, or sum of dyads, as the most general type of linear vector operator. The dyadic is useful not only in the treatment of the linear vector transformations or strains, but also as a mere formal product (or sum of products) which can later be converted into such determinate products as the outer and inner products of our analysis. We shall outline very briefly the form taken by the theory of dyadics in the vector analysis which we employed.[65]

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, . . . are 1-vectors, then the product expressed by the mere juxtaposition of $\mathbf{a}$ and $\mathbf{b}$, namely, $\mathbf{ab}$ is called a dyad. The sum of two or more such dyads is called a dyadic, and any such dyadic in an n-dimensional space can be reduced to the sum of n dyads. As the dyad is in part defined by the assumption of the distributive law, every dyadic in four dimensional space may be expressed as a block of sixteen terms analogous to a matrix. Such an expansion is of great convenience when the individual vectors are expressed in terms of coordinate vectors. Thus,

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[64] It should, however, be noted that there is nothing in electromagnetics corresponding to the vector of extended momentum of energy moving with the velocity of light. It is, furthermore, to be noted that while the radiation fields produced by the acceleration of two electrons, whether of the same or opposite sign, due to their interaction, are cumulative, that produced by the acceleration of two material particles, due to their gravitational attraction, must tend to compensate one another. (Cf. the paper of D. L. Webster, These Proceedings, **47**, 569, 1912.)

[65] For further developments we refer to Gibbs's work as set forth in his Scientific Papers, **2**, in the Gibbs-Wilson text on Vector Analysis, and in Wilson's "On the theory of double products and strains in hyperspace," Trans. Conn. Acad., **14**, 1.

$$
\begin{aligned}
& a_{11}\mathbf{k}_1\mathbf{k}_1 + a_{12}\mathbf{k}_1\mathbf{k}_2 + a_{13}\mathbf{k}_1\mathbf{k}_3 + a_{14}\mathbf{k}_1\mathbf{k}_4 \\
+ \; & a_{21}\mathbf{k}_2\mathbf{k}_1 + a_{22}\mathbf{k}_2\mathbf{k}_2 + a_{23}\mathbf{k}_2\mathbf{k}_3 + a_{24}\mathbf{k}_2\mathbf{k}_4 \\
+ \; & a_{31}\mathbf{k}_3\mathbf{k}_1 + a_{32}\mathbf{k}_3\mathbf{k}_2 + a_{33}\mathbf{k}_3\mathbf{k}_3 + a_{34}\mathbf{k}_3\mathbf{k}_4 \\
+ \; & a_{41}\mathbf{k}_4\mathbf{k}_1 + a_{42}\mathbf{k}_4\mathbf{k}_2 + a_{43}\mathbf{k}_4\mathbf{k}_2 + a_{44}\mathbf{k}_4\mathbf{k}_4.
\end{aligned}
$$

The product of a vector **a** and a dyad **bc** is expressed and defined as

$$\mathbf{a}\cdot\mathbf{bc} = \mathbf{a}\cdot(\mathbf{bc}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c},$$

It is a 1-vector along **c**. Similarly $\mathbf{ab}\cdot\mathbf{c} = (\mathbf{ab})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$. The product of a vector into any dyadic follows from the distributive law.

The product of two dyads is expressed and defined as follows.

$$\mathbf{ab}\cdot\mathbf{cd} = (\mathbf{ab})\cdot(\mathbf{cd}) = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})\mathbf{d} = (\mathbf{b}\cdot\mathbf{c})\mathbf{ad}.$$

It is another dyad. The product of two dyadics then follows from the distributive law, and is therefore a dyadic.

Since the dyad product is obtained without implying any relation between the sixteen units $\mathbf{k}_i\mathbf{k}_j$, it is the most general product and comprises within itself the more special products which we have designated as the inner and outer products and which we may obtain from it by inserting the special sign of multiplication corresponding to these products, thus giving respectively a scalar or a 2-vector. Hence from any dyadic a scalar or a 2-vector may be obtained by converting each dyad into an inner or outer product. This method was employed in computing $\Diamond\cdot\mathbf{p}$ and $\Diamond\times\mathbf{p}$ in § 43 and § 44.

A dyadic is said to selfconjugate when for all the coefficients $a_{ij} = a_{ji}$, and anti-selfconjugate when for all the coefficients $a_{ij} = -a_{ji}$. The latter can have no terms in the main diagonal, and therefore has but six degrees of freedom, whereas the selfconjugate dyadic has ten.[66] Except for sign the anti-selfconjugate dyadic not only determines, but conversely is determined by, a 2-vector of the form

$$a_{12}\mathbf{k}_{12} + a_{13}\mathbf{k}_{13} + a_{14}\mathbf{k}_{14} + a_{23}\mathbf{k}_{23} + a_{24}\mathbf{k}_{24} + a_{34}\mathbf{k}_{34},$$

where $a_{12}, \ldots$ are the coefficients of $\mathbf{k}_1\mathbf{k}_2, \ldots$ in the expanded form of the dyadic. This 2-vector is one half the 2-vector obtained by inserting the sign of outer multiplication in the dyads constituting the dyadic.

---

[66] Any dyadic may be written as the sum of two dyadics one of which is selfconjugate, the other anti-selfconjugate.

If Φ is any dyadic, then we have seen that $\mathbf{a}\cdot\Phi$ is another 1-vector. In general $\mathbf{a}\cdot\Phi$ is not equal to $\Phi\cdot\mathbf{a}$. If, however, Φ is selfconjugate, $\mathbf{a}\cdot\Phi = \Phi\cdot\mathbf{a}$; and if Φ is anti-selfconjugate $\mathbf{a}\cdot\Phi = -\Phi\cdot\mathbf{a}$. Hence it may readily be shown that an anti-selfconjugate dyadic turns a vector into a perpendicular vector.

The dyadic which turns a vector into itself is called the idemfactor I. Thus

$$\mathbf{a}\cdot\mathrm{I} = \mathrm{I}\cdot\mathbf{a} = \mathbf{a}; \tag{154}$$

for I is selfconjugate, and when expanded in terms of chosen coordinate vectors is,[67] in the non-Euclidean geometry which we are discussing,

$$\mathrm{I} = \mathbf{k}_1\mathbf{k}_1 + \mathbf{k}_2\mathbf{k}_2 + \mathbf{k}_3\mathbf{k}_3 - \mathbf{k}_4\mathbf{k}_4.$$

62. We could now proceed to develop the theory of dyadics involving vectors of any dimensionalities and their products with each other and with vectors of various dimensionalities. In general if α, β, γ are vectors of any dimensionalities the dyad $\beta\gamma$ may be defined in terms of our inner product by the equation $\alpha\cdot(\beta\gamma) = (\alpha\cdot\beta)\gamma$. This product is itself a dyad unless α, β are of the same dimensionality. Such a discussion, however, would carry us further than is necessary for our present purpose, and we shall therefore consider chiefly one case, which has acquired particular importance through the work of Minkowski.

If $\mathbf{r}$ is any 1-vector, and $\mathbf{A}$ any 2-vector, then the product

$$\mathbf{r}' = \mathbf{r}\cdot\mathbf{A}$$

is a linear vector function of $\mathbf{r}$. It is evident therefore that this multiplication by $\mathbf{A}$ is equivalent to a multiplication by some dyadic Ω. Let us find the relation between this dyadic Ω and $\mathbf{A}$.

If Φ is any dyadic (made up of 1-vectors), we may define the products $\Phi\cdot\mathbf{A}$ and $\mathbf{A}\cdot\Phi$ by first defining the products,

$$(\mathbf{ab})\cdot\mathbf{A} = \mathbf{a}(\mathbf{b}\cdot\mathbf{A}), \qquad \mathbf{A}\cdot(\mathbf{ab}) = (\mathbf{A}\cdot\mathbf{a})\,\mathbf{b},$$

---

[67] As a matrix the idemfactor would be written

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \quad \text{instead of} \quad \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix};$$

and the laws of multiplication of matrices would be modified. It is possible, however, to keep the ordinary theory of matrices by the introduction of imaginaries, as Minkowski does.

and then applying the distributive law. The products $\mathbf{A}\cdot\Phi$ and $\Phi\cdot\mathbf{A}$ are therefore themselves dyadics of the same type as Φ. If in place of Φ we use the idemfactor I, then it is easily shown that

$$\mathrm{I}\cdot\mathbf{A}\ (= -\mathbf{A}\cdot\mathrm{I})$$

is the anti-selfconjugate dyadic which is determined by the 2-vector **A**.

$$\left.\begin{aligned}\Omega = \mathrm{I}\cdot\mathbf{A} = \qquad\qquad & \qquad\qquad - A_{12}\mathbf{k}_1\mathbf{k}_2 - A_{13}\mathbf{k}_1\mathbf{k}_3 - A_{14}\mathbf{k}_1\mathbf{k}_4 \\ & + A_{12}\mathbf{k}_2\mathbf{k}_1 \qquad\qquad - A_{23}\mathbf{k}_2\mathbf{k}_3 - A_{24}\mathbf{k}_2\mathbf{k}_4 \\ & + A_{13}\mathbf{k}_3\mathbf{k}_1 + A_{23}\mathbf{k}_3\mathbf{k}_2 \qquad\qquad - A_{34}\mathbf{k}_3\mathbf{k}_4 \\ & + A_{14}\mathbf{k}_4\mathbf{k}_1 + A_{24}\mathbf{k}_4\mathbf{k}_2 + A_{34}\mathbf{k}_4\mathbf{k}_3\end{aligned}\right\} \tag{155}$$

If we denote by $\Omega_\times$ the 2-vector obtained by inserting the cross in the dyads of Ω, we have $\Omega_\times = (\mathrm{I}\cdot\mathbf{A})_\times = -2\mathbf{A}$.

It is this relation between 2-vectors and linear vector functions or dyadics which enables Minkowski to replace a 2-vector by an anti-selfconjugate (or alternating) matrix and vice versa.

If Ω and Ω' are the two dyadics obtained from the two 2-vectors **A** and **A**′, we may form the product $\Omega\cdot\Omega'$. (This is the product *fF* of Minkowski). We can then write

$$(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}' = (\mathbf{r}\cdot\Omega)\cdot\Omega' = \mathbf{r}\cdot(\Omega\cdot\Omega'). \tag{156}$$

We employed (§ 57) the selfconjugate dyadic $\Omega\cdot\Omega = (\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A})$, and another dyadic $\frac{1}{2}(\Omega\cdot\Omega + \Omega^*\cdot\Omega^*)$, where Ω^* was defined as $\Omega^* = \mathrm{I}\cdot\mathbf{A}^*$. This dyadic corresponds to the matrix *S* of Minkowski,[68] and may be regarded as the dyadic representing stress in four dimensional space.

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[68] The expression $(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}'$ may be transformed by (38).

$$(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}' = -\mathbf{r}(\mathbf{A}\cdot\mathbf{A}') + \mathbf{A}\cdot(\mathbf{r}\times\mathbf{A}').$$

As $\mathbf{A}\cdot(\mathbf{r}\times\mathbf{A}')$ is a 1-vector, the complement of its complement is itself, by (26). By rules (30) and (24)

$$[\mathbf{A}\cdot(\mathbf{r}\times\mathbf{A}')]^{**} = [\mathbf{A}\times(\mathbf{r}\times\mathbf{A}')^*]^* = [(\mathbf{r}\cdot\mathbf{A}'^*)\times\mathbf{A}]^* = (\mathbf{r}\cdot\mathbf{A}'^*)\cdot\mathbf{A}^*.$$

Hence we obtain the important relation

$$(\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}' = -\mathbf{r}(\mathbf{A}\cdot\mathbf{A}') + (\mathbf{r}\cdot\mathbf{A}'^*)\cdot\mathbf{A}^*.$$

By introducing dyadics and canceling the vector **r**, we have

$$(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') = -(\mathbf{A}\cdot\mathbf{A})\mathrm{I} + (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*).$$

If we set

$$\Psi = \tfrac{1}{2}[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') + (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*)],$$

we may write

$$(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') = \Psi - \tfrac{1}{2}(\mathbf{A}\cdot\mathbf{A}')\mathrm{I}, \qquad (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*) = \Psi + \tfrac{1}{2}(\mathbf{A}\cdot\mathbf{A}')\mathrm{I}.$$

The dyadic Ψ is precisely the matrix *S* of Minkowski.

The transformation $\mathbf{r}' = \mathbf{r}\cdot\mathbf{A}$, where $\mathbf{A}$ is a uniplanar 2-vector, can be regarded geometrically as an annihilation of that part of $\mathbf{r}$ which is perpendicular to $\mathbf{A}$, and a replacing of the component of $\mathbf{r}$ in $\mathbf{A}$ by a perpendicular vector magnified in the ratio of A to 1. The transformation $\mathbf{r}' = (\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}$ therefore annihilates components perpendicular to $\mathbf{A}$, and reverses components in $\mathbf{A}$, multiplying them further by $\mathbf{A}\cdot\mathbf{A}$. Hence if $\mathbf{A}$ is a (γ)-plane, the transformation in that plane is rotation through a straight angle combined with a stretch as $A^2:1$; whereas if $\mathbf{A}$ is a (δ)-plane, the transformation is one of stretching only, as $\mathbf{A}\cdot\mathbf{A}$ is negative.

In case $\mathbf{A}$ is biplanar we may resolve it into its two completely perpendicular parts, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{B}$ is a (γ)-vector and $\mathbf{C}$ a (δ)-vector. Then the equation

$$\mathbf{r}' = (\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A} = (\mathbf{r}\cdot\mathbf{B})\cdot\mathbf{B} + (\mathbf{r}\cdot\mathbf{C})\cdot\mathbf{C}$$

holds by virtue of the fact that $\mathbf{r}\cdot\mathbf{B}$ is perpendicular to $\mathbf{C}$, and $\mathbf{r}\cdot\mathbf{C}$ perpendicular to $\mathbf{B}$. Hence the transformation $\mathbf{r}' = (\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A}$ consists of rotation through a straight angle and stretching in the ratio $B^2:1$ for components along $\mathbf{B}$, and of stretching alone in the ratio $C^2:1$ for components along $\mathbf{C}$.

The transformation $\mathbf{r}' = (\mathbf{r}\cdot\mathbf{A})\cdot\mathbf{A} + (\mathbf{r}\cdot\mathbf{A}^*)\cdot\mathbf{A}^*$ is now readily seen to be a stretching of components along $\mathbf{B}$ or $\mathbf{C}$ in the ratio $(B^2 + C^2):1$ combined with a reversal of the direction of the components along $\mathbf{B}$. If this transformation were repeated, the result would be to stretch all vectors in space in the ratio $(B^2 + C^2)^2:1$. But

$$(B^2 + C^2)^2 = (B^2 - C^2)^2 + 4B^2C^2 = (\mathbf{A}\cdot\mathbf{A})^2 + (\mathbf{A}\cdot\mathbf{A}^*)^2.$$

Hence the square of $\frac{1}{2}(\Omega\cdot\Omega + \Omega^*\cdot\Omega^*)$ is $\frac{1}{4}[(\mathbf{A}\cdot\mathbf{A})^2 + (\mathbf{A}\cdot\mathbf{A}^*)^2]\ \mathbf{I}$, a multiple of the idemfactor. This is the geometric interpretation of a result obtained analytically by Minkowski.

63. From the definition (48) of the differentiating operator $\diamondsuit$,

$$d\mathbf{f} = d\mathbf{r}\cdot\diamondsuit\mathbf{f},$$

it follows that the expression $\diamondsuit\mathbf{f}$, where $\mathbf{f}$ is a 1-vector, is a dyadic. This definition may frequently be applied directly and with ease to determining the dyadic $\diamondsuit\mathbf{f}$, and renders unnecessary the expansion of $\diamondsuit\mathbf{f}$ in terms of its components. For if the value of $d\mathbf{f}$ for four independent displacements $d\mathbf{r}$ can be found, the dyadic is thereby completely determined, and in some cases can immediately be written down by inspection. This was the method pursued in § 44. The dyadic itself, however, was not then desired except for the purpose of deriving the scalar $\Diamond\cdot\mathbf{f}$ and the 2-vector $\Diamond\times\mathbf{f}$, which are functions of it.

By means of the same defining equation the operator $\Diamond$ may be applied to 2-vector functions of position. The result $\Diamond\mathbf{F}$ is then a dyadic in which the first vectors of the dyads are 1-vectors and the second vectors 2-vectors. If written out in terms of the coordinate unit vectors, such a dyadic would consist of twenty-four terms, each of the type $\mathbf{k}_i\mathbf{k}_{jk}$, $j \neq k$. By inserting the dot or cross, the 1-vector $\Diamond\cdot\mathbf{F}$ and the 3-vector $\Diamond\times\mathbf{F}$ are immediately found. In case the 2-vector $\mathbf{F}$ is given as a product $\mathbf{f}\times\mathbf{g}$ of two 1-vectors, the dyadic $\Diamond\mathbf{F}$ may be obtained directly by means of the rules of differentiation in terms of the dyadics $\Diamond\mathbf{f}$ and $\Diamond\mathbf{g}$. For

$$d\mathbf{r}\cdot\Diamond\mathbf{F} = d\mathbf{F} = d(\mathbf{f}\times\mathbf{g}) = d\mathbf{f}\times\mathbf{g} + d\mathbf{f}\times d\mathbf{g} = \mathbf{f}\times\mathbf{g} - d\mathbf{g}\times\mathbf{f},$$

$$d\mathbf{r}\cdot\Diamond\mathbf{F} = d\mathbf{r}\cdot\Diamond\mathbf{f}\times\mathbf{g} - d\mathbf{r}\cdot\Diamond\mathbf{g}\times\mathbf{f},$$

$$\Diamond\mathbf{F} = \Diamond\mathbf{f}\times\mathbf{g} - \Diamond\mathbf{g}\times\mathbf{f}.$$

It was such analysis which was used in § 44. It illustrates strikingly the great advantage of the symbol $\Diamond$ over such symbols as Div, Rot, Grad, and 𝔇𝔦𝔳.

If Ψ is a dyadic function of position, the equation $d\mathbf{r}\cdot\Diamond\Psi = d\Psi$ may be used to define $\Diamond\Psi$, which is a triadic, that is, a sum of formal products of which each contains three vectors juxtaposed without any sign of multiplication. By interposing a dot between the first two of the three vectors in the triads, we find the 1-vector $\Diamond\cdot\Psi$. The expression $\Diamond\cdot\Psi$ corresponds to what Minkowski calls lor Ψ, where Ψ is for him a matrix.

We may compute the expression $\Diamond\Psi$ in the case where

$$\Psi = \tfrac{1}{2}[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') + (\mathrm{I}\cdot\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*)]. \qquad (157)$$

First we write

$$d\mathbf{r}\cdot\Diamond\,[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')] = d[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')]$$
$$= [d(\mathrm{I}\cdot\mathbf{A})]\cdot(\mathrm{I}\cdot\mathbf{A}') + (\mathrm{I}\cdot\mathbf{A})\cdot d(\mathrm{I}\cdot\mathbf{A}').$$

The second term may be transformed so that the differential comes to the front. For by the equation found in the previous footnote,

$$(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot d\mathbf{A}') = -\mathbf{A}\cdot d\mathbf{A}'\mathrm{I} + (\mathrm{I}\cdot d\mathbf{A}'^*)\cdot(\mathrm{I}\cdot\mathbf{A}^*).$$

Hence

$$d[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')] = -(d\mathbf{A}\cdot\mathrm{I})\cdot(\mathrm{I}\cdot\mathbf{A}') - (d\mathbf{A}'^*\cdot\mathrm{I})\cdot(\mathrm{I}\cdot\mathbf{A}^*) - d\mathbf{A}'\cdot\mathbf{A}\mathrm{I}.$$

Now

$$(d\mathbf{A}\cdot\mathrm{I})\cdot(\mathrm{I}\cdot\mathbf{A}') = d\mathbf{A}\cdot(\mathrm{I}\cdot\mathrm{I}\cdot\mathbf{A}') = d\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}').$$

Hence

$$d\mathbf{r}\cdot\Diamond[(\mathrm{I}\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')]$$
$$= (d\mathbf{r}\cdot\Diamond\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}') - (d\mathbf{r}\cdot\Diamond\mathbf{A}'^{*})\cdot(\mathrm{I}\cdot\mathbf{A}^{*}) - d\mathbf{r}\cdot\Diamond\mathbf{A}'\cdot\mathbf{A}\mathrm{I}.$$

Hence finally

$$2\Diamond\Psi = -\Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}') - \Diamond\mathbf{A}'^{*}\cdot(\mathrm{I}\cdot\mathbf{A}^{*}) - \Diamond\mathbf{A}'\cdot\mathbf{A}\mathrm{I}$$
$$- \Diamond\mathbf{A}'^{*}\cdot(\mathrm{I}\cdot\mathbf{A}^{*}) - \Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}') + \Diamond\mathbf{A}\cdot\mathbf{A}'\mathrm{I}.$$

If the expression $\Diamond\cdot\Psi$ is desired, care must be exercised to insert the dot between the first two vectors of each triad. Hence [69]

$$2\Diamond\cdot\Psi = 2(\Diamond\cdot\mathbf{A})\cdot\mathbf{A}' + 2(\Diamond\cdot\mathbf{A}'^{*})\cdot\mathbf{A}^{*} - \Diamond\mathbf{A}'\cdot\mathbf{A} + \Diamond\mathbf{A}\cdot\mathbf{A}',$$
$$\Diamond\cdot\Psi = (\Diamond\cdot\mathbf{A})\cdot\mathbf{A}' + (\Diamond\cdot\mathbf{A}'^{*})\cdot\mathbf{A}^{*} + \tfrac{1}{2}(\Diamond\mathbf{A}\cdot\mathbf{A}' - \Diamond\mathbf{A}'\cdot\mathbf{A}). \quad (158)$$

Some Projective Geometry, and Trigonometry.

64. We may discuss very briefly the relations between our non-Euclidean measure of angle and the projective measure as determined by logarithms of cross-ratios. Let us consider Figure 30 first as a Euclidean and second as a non-Euclidean diagram. The two fixed lines α, β are drawn so that they are perpendicular from the Euclidean point of view. The initial line from which angles are measured is taken as the bisector of one of the right angles; this line and its perpendicular through the origin will be taken as axes of x and y. The pseudo-circle appears as a rectangular hyperbola with the equation $x^2 - y^2 = 1$. The angle between the initial line and any radius in the pseudo-circle in Euclidean measure will be called θ, and $\tan\theta = y/x$. Now in non-Euclidean measure, if this angle be called ϕ, we have seen that $\tanh\phi = y/x$. Hence we have the relation

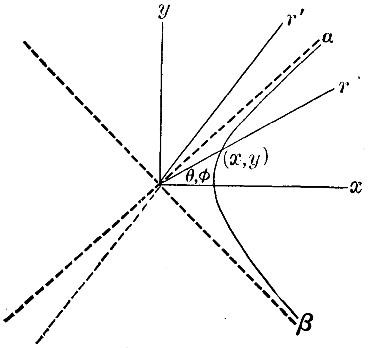

FIGURE 30.

$$\tan\theta = \tanh\phi.$$

---

[69] The form $\Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}')$ may be written as a sum of triads of the type $\mathbf{aA}\cdot(\mathbf{ef})$ or $\mathbf{a}(\mathbf{A}\cdot\mathbf{e})\,\mathbf{f}$. Now by (35), $\mathbf{a}\cdot(\mathbf{A}\cdot\mathbf{e}) = -(\mathbf{a}\cdot\mathbf{A})\cdot\mathbf{e}$. Hence the insertion of a dot in $\Diamond\mathbf{A}\cdot(\mathrm{I}\cdot\mathbf{A}')$ gives $-(\Diamond\cdot\mathbf{A})\cdot(\mathrm{I}\cdot\mathbf{A}')$ or $-(\Diamond\cdot\mathbf{A})\cdot\mathbf{A}'$. In the form $\Diamond\mathbf{A}\cdot\mathbf{A}'\mathrm{I}$, the dot goes between $\Diamond$ and I, since $\mathbf{A}\cdot\mathbf{A}'$ is a scalar. But as I is the idemfactor, we have simply $\Diamond\mathbf{A}\cdot\mathbf{A}'$ as the result.

The cross-ratio formed by the four lines, x, r, α, β is

$$\lambda = \frac{\sin\angle(\beta, r)\,\sin\angle(x, \alpha)}{\sin\angle(r, \alpha)\,\sin\angle(\beta, x)},$$

where the angles are measured in Euclidean fashion. Hence

$$\lambda = \frac{\sin\left(\frac{\pi}{4} + \theta\right)}{\sin\left(\frac{\pi}{4} - \theta\right)} = \frac{1 + \tan\theta}{1 - \tan\theta} = \frac{1 + \tanh\phi}{1 - \tanh\phi} = e^{2\varphi}.$$

Or

$$\phi = \tfrac{1}{2}\log\lambda.$$

Hence the non-Euclidean angle is measured by one-half the logarithm of the cross-ratio of four rays. Although the Euclidean point of view has been adopted for simplicity, the final result, depending as it does only on the cross-ratio, is projective; it is therefore independent of the particular assumptions that the rays α and β are perpendicular and that the initial line bisects the angle between them.

Consider next a ray r' such that in the Euclidean sense

$$\angle(\alpha, r') = \angle(r, \alpha).$$

(In the non-Euclidean sense r and r' are perpendicular). In forming the cross-ratio it is evident that $\lambda' = -\lambda$. Hence for the non-Euclidean angle ϕ' between x and r'

$$\phi' = \tfrac{1}{2}\log\lambda' = \tfrac{1}{2}\log(-\lambda) = \phi + \tfrac{1}{2}\log(-1).$$

Hence

$$\phi' = \phi \pm \tfrac{1}{2}\pi i.$$

The angle $\phi' - \phi$, that is, the angle between two lines perpendicular in the non-Euclidean sense is therefore $\pm \frac{1}{2}\pi i$. This result also is projective and independent of our special assumptions. It is only natural that the angle between two lines in different classes should appear as a complex number, owing to the fact that it is impossible to rotate one line into the other.

In setting up a projective measure of angle by means of cross-ratios, it is customary among mathematicians to define the angle as

$$\phi = \frac{1}{2\sqrt{-1}}\log\lambda,$$

where the logarithm of the cross-ratio is divided by $2i$ instead of by 2 as above. The choice of the divisor $2i$ is due to the desire to have the angle real when the fixed lines are conjugate imaginary lines and to have the total angle about a point equal to 2π as in Euclidean geometry; this is not, however, in any way suggested by projective geometry. In our non-Euclidean geometry, where we have taken a different set of postulates for rotation, the real divisor 2 is more natural. We have seen that from the point of view of the postulates of translation or the parallel transformation our geometry and the ordinary Euclidean geometry fall into one class, while such geometries as the Lobatchewskian and the Riemannian belong to another class. With respect to the postulates of rotation, however, the Euclidean and most of the non-Euclidean geometries which have been studied lie in one class, to which our geometry does not belong. The methods of projective geometry are applicable to all these classes.

If the ray r is perpendicular to the rays r' and r'', the latter two being in the same line but oppositely directed, it is evident that we must choose arbitrarily the sign of the angle $\pm \frac{1}{2}\pi i$ between r and r'; but we shall assume that if the sign of the angle rr' has been determined the sign of the angle rr'' will be the same. Thus the angle $r'r''$ is zero. This means that a pair of intersecting lines determine but one angle except for sign; thus any angle is identical, except for sign, with its supplement.

The angle from a line to a second line and the angle from the first line to the perpendicular to the second will be called complementary. The complement of a real angle is a complex angle, and vice versa.

65. Hitherto we have chosen to avoid the use of the term distance, and have used the word interval to represent a positive number expressing the measure of length. If r is a line drawn from the origin, the interval of r has been defined as $\sqrt{x^2 - y^2}$ or $\sqrt{y^2 - x^2}$ according as x is greater than y or y greater than x. This was done to avoid altogether the use of imaginaries. We might, however, have defined distance as

$$\int \pm \sqrt{dx^2 - dy^2},$$

where x is, for example, measured along a (γ)-line, y along a perpendicular (δ)-line. Then every (γ)-line would have a real, and every (δ)-line an imaginary distance. In this case it would be convenient to consider the distance along any vector AB as the negative of the distance along BA. The distance along any singular line is zero.

The preceding ideas can be used to give new definitions of the inner and outer products of two vectors. Namely,

$$\mathbf{a}\cdot\mathbf{b} = \text{distance of } \mathbf{a} \text{ times distance of } \mathbf{b} \text{ times } \cosh \angle\ (\mathbf{a}, \mathbf{b}),$$

$$\mathbf{a}\times\mathbf{b} = \text{distance of } \mathbf{a} \text{ times distance of } \mathbf{b} \text{ times } \sinh \angle\ (\mathbf{a}, \mathbf{b}),$$

it being understood that the latter quantity is not a scalar but a pseudo-scalar. If **a** and **b** are vectors of the same class the angles are real, and the equations are essentially identical with those which have been previously derived. If **a** and **b** are (δ)-vectors the distances are purely imaginary and the product $\mathbf{a}\cdot\mathbf{b}$ is negative if the vectors issue into the same "quadrant." If **a** and **b** are of different classes, and the angle between them complex, we may use in place of these complex angles their complementary real angles by the aid of the familiar formulas

$$\cosh(\phi + \tfrac{1}{2}\pi i) = i \sinh\phi, \qquad \sinh(\phi + \tfrac{1}{2}\pi i) = i \cosh\phi.$$

Massachusetts Institute of Technology,
Boston, Mass., *May*, 1912.

TABLE OF NOTATIONS.

General Symbols.
- 1-vectors, lower case Clarendons, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,...;
  - their magnitudes, corresponding Italic, a, b, c,...;
  - their components (algebraic magnitudes), a_1, a_2, a_3, a_4, etc.;
  - their (vector) space components, $\mathbf{a}_s$, $\mathbf{b}_s$, $\mathbf{c}_s$,....
  - $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, unit coordinate space vectors;
  - $\mathbf{k}_4$, unit coordinate time vector.
- 2-vectors, Clarendon capitals, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$,...;
  - their magnitudes, corresponding Italic, A, B, C,...;
  - their components, A_{12}, A_{23},..., A_{34}; etc.;
  - $\mathbf{k}_{12}$, $\mathbf{k}_{23}$,..., k_{34}, unit coordinate 2-vectors.
- 3-vectors, Tudor black capitals, $\boldsymbol{\mathfrak{A}}$, $\boldsymbol{\mathfrak{B}}$, $\boldsymbol{\mathfrak{C}}$,...;
  - their magnitudes, corresponding German, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$,...;
  - their components, A_{234},..., A_{123};
  - $\mathbf{k}_{234}$,..., $\mathbf{k}_{123}$, unit coordinate 3-vectors (the last, "space").
- unit pseudo-scalar, $\mathbf{k}_{1234}$.
- sign of the outer product, small cross, $\times$.
- sign of the inner product, heavy dot, $\cdot$.
- sign of the complement, asterisk, $\mathbf{a}^*$, $\mathbf{A}^*$,....
- three-dimensional differentiating operator, del, ∇.
- four-dimensional differentiating operator, quad, $\diamondsuit$.
- dyadics, Greek capitals, Φ,... (idemfactor, I).

Special Symbols (non-vectorial).
α, β, singular lines (§ 9).
γ, spacial lines; δ, temporal lines (§ 9, 37).
ϵ, electric charge (§ 48).
μ, material density (§ 45); μ_0, density under no relative motion.
ρ, electric density (§ 54); ρ_0, density under no relative motion.
φ, electric scalar potential (§ 48).
m, mass; m_0, mass under no relative motion.
t, time (also x_4).
u, v, velocities.
x, y, z, space coordinates (also x_1, x_2, x_3).
I, idemfactor.
L, Lagrangian function (§ 56).
R, a perpendicular interval (§ 43).

Special Symbols (vectorial).
$\mathbf{a}$, (three-dimensional) ordinary vector potential (§ 48).
$\mathbf{b}$, a special four-dimensional "radiation field" (§ 53).
$\mathbf{c}$, extended curvature (§ 22, 35).
$\mathbf{e}$, (three-dimensional) electric force (§ 49, 50).
$\mathbf{f}$, (three-dimensional) mechanical force (§ 35).
$\mathbf{g}$, as in $d\mathbf{g}$, special vector of extended momentum (§ 47).
$\mathbf{h}$, (three-dimensional) magnetic force (§ 49, 50).
$\mathbf{l}$, extended light-vector, singular ray (§ 43).
$\mathbf{m}$, extended (four-dimensional) vector potential (§ 48, 55).
$\mathbf{n}$, unit normal to (δ)-curve (§ 43).
$\mathbf{p}$, geometric potential vector (§ 43).
$\mathbf{q}$, vector of extended electric current density (§ 54).
$\mathbf{r}$, four-dimensional radius vector.
$\mathbf{s}$, as in $d\mathbf{s}$, vector element of arc.
$\mathbf{v}$, (three-dimensional) velocity (§ 43).
$\mathbf{w}$, unit tangent to (δ)-curve.
$\mathbf{E}$, electric 2-vector (§ 49).
$\mathbf{H}$, magnetic 2-vector (§ 49).
$\mathbf{M}$, electromagnetic 2-vector (§ 48).
$\mathbf{P}$, geometric 2-vector field (§ 43).
$\mathbf{S}$, as in $d\mathbf{S}$, element of (two-dimensional) surface.
$\mathfrak{S}$, as in $d\mathfrak{S}$, element of three-dimensional volume.
Σ, as in $d\Sigma$, element of four-dimensional volume.

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